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ON AXIALLY SYMMETRIC FLOW AND THE METHOD OF GENERALIZED ELFCTROSTATICS*

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1. Introduction. In his recent work in Generalized Axially Symmetric Potential Theory, A. Weinstein [1] has pointed out that the flow about an axially symmetric body in ordinary three-dimensional space may be obtained from the electrostatic potential of the five-dimensional body of revolution possessing the same meridian profile. This method of solution is referred to as the method of Generalized Electrostatics. It has been recently used by L. E. Payne and A. Weinstein [2] in deriving a relationship between capacity and virtual mass and has been employed by A. Weinstein [3] in solving certain torsion problems.

In this paper the method of Generalized Electrostatics is used to obtain the flow about a spindle and a lens. The flow about a spindle seems not to have been treated in the literature despite its obvious importance. The lens problem however was treated in 1947 by M. Shiffman and D. C. Spencer [4] who applied an ingenious and difficult procedure involving the method of images in a multi-sheeted Riemann-Sommerfeld space. The solution given in this paper is a straightforward generalization of results given already in 1868 by F. G. Mehler [5]. Our formulas are considerably simpler than those of Shiffman and Spencer but there is no obvious computational method of showing that their solution may be reduced to ours or vice versa. However, the identity of these two solutions is guaranteed by a uniqueness theorem. In an oral communique Professor D. C. Spencer has pointed out the fact that the problem of the spindle would be difficult if not impossible to solve by the method of images in a Riemann-Sommerfeld space. It will be seen that generally speaking little difficulty is encountered in extending to an arbitrary odd-dimensional space the known solutions of three-dimensional electrostatics problems. Hence by the method of Generalized Electrostatics we obtain the flows about axially symmetric profiles almost immediately from the electrostatic solutions.

In this paper we are concerned chiefly with bodies of revolution in three- and five-dimensional spaces; but since spaces of other dimensions have various applications we shall first obtain the solution in a general n-dimensional space. The ordinary three-dimensional terminology will be retained throughout. Later we shall assign the particular value to n which is demanded by the physical problem.

2. Basic Equations. We shall restrict ourselves in this paper to profiles of revolution

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in a uniform stream. We shall assume further that the fluid is incompressible and that at infinity it is travelling at uniform velocity U parallel to the axis of symmetry. Let xy be the meridian plane of an n-dimensional body which is symmetric about the x-axis. We shall define all functions in the meridian half plane $y \geqq 0$.

An axially symmetric potential function $\varphi\{n\}$ in a space of n-dimensions is a solution of the partial differential equation

$$\frac{\partial}{\partial x}\left(y^{n-2}\frac{\partial \varphi\{n\}}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^{n-2}\frac{\partial \varphi\{n\}}{\partial y}\right) = 0, \qquad n \geqq 2. \tag{1}$$

The associated stream function $\psi\{n\}$ is defined with the aid of the generalized Stokes-Beltrami equations

$$y^{n-2}\frac{\partial \varphi\{n\}}{\partial x} = \frac{\partial \psi\{n\}}{\partial y}, \qquad y^{n-2}\frac{\partial \varphi\{n\}}{\partial y} = -\frac{\partial \psi\{n\}}{\partial x}. \tag{2}$$

Let $\Psi\{n\} = Uy^{n-1}(n-1)^{-1} - \psi\{n\}$ be the stream function describing our flow. We assume $\Psi\{n\}$ to vanish on the profile and along the x-axis.* We make use of a correspondance relationship [1], namely $\psi\{n\} = Uy^{n-1}(n-1)^{-1}\varphi\{n+2\}$, to obtain the fundamental equation

$$\Psi\{n\} = Uy^{n-1}(n-1)^{-1}(1 - \varphi\{n+2\}). \tag{3}$$

This equation relates the stream function $\Psi\{n\}$ for an n-dimensional body of revolution B to an electrostatic potential function $\varphi\{n+2\}$ in a space of $n+2$ dimensions. The potential $\varphi\{n+2\}$ assumes the value unity on the profile boundary and vanishes at infinity.

We may by introducing the substitution $\chi\{n\} = y^{(n-2)/2}\varphi\{n\}$ reduce Eq. (1) to the form

$$\nabla^2\chi - [(n-2)(n-4)/4y^2]\chi = 0 \tag{4}$$

where ∇^2 denotes the Laplacian operator. It is obvious that for $n = 2$ or $n = 4$, χ is harmonic. C. Snow [6] in a discussion of non-axially symmetric potential problems in three dimensions has treated solutions of Eq. (4) for odd values of n.

Under a transformation $x + iy = z = f(\zeta) = f(\xi + i\eta)$ Eq. (4) takes the form

$$\chi_{\xi\xi} + \chi_{\eta\eta} - [(n-2)(n-4)/4h^2y^2]\chi = 0 \tag{5}$$

where $h^2 = |f'(\zeta)|^{-2}$. Similarly under such a transformation Eq. (1) becomes

$$\frac{\partial}{\partial \xi}\left(y^{n-2}\frac{\partial \varphi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(y^{n-2}\frac{\partial \varphi}{\partial \eta}\right) = 0. \tag{6}$$

If y is of the form $f(\xi) \cdot g(\eta)$ the solution of (6) is readily obtained by separation of variables. If on the other hand $(h^2y^2)^{-1} = f_1(\xi) + g_1(\eta)$ as in the cases considered here, we find it more convenient to use Eq. (5).

Payne and Weinstein [2] have derived a relationship between the n-dimensional virtual mass $M\{n\}$ (uniform flow in the x-direction) and the $n+2$-dimensional capacity $C\{n+2\}$ which for $n = 3$ is given by

$$M\{3\} + V\{3\} = (2\pi/3)C\{5\}. \tag{7}$$

*This implies that the profile and the x-axis form a single streamline.

In (7) $V\{3\}$ denotes the volume of the body of revolution (the density of the fluid has been taken as unity). The capacity $C\{n\}$ is determined from the equation

$$C\{n\} = (n - 2) \lim_{r \to \infty} r^{n-2} \varphi\{n\}, \qquad r^2 = x^2 + y^2 \tag{8}$$

where $\varphi\{n\}$ is the electrostatic potential of the body. We shall make use of (7) and (8) in determining the virtual mass of the spindle and the lens.

3. Flow About a Spindle. A curve $\xi = \xi_0$ in dipolar coordinates defines the profile of a a spindle. The dipolar transformation is given by

$$x + iy = ic \cot \tfrac{1}{2}(\xi + i\eta) \tag{9}$$

where c is a positive constant. The range of coordinates is chosen as $-\infty < \eta < +\infty$, $0 < \xi \leq \pi$. The boundary of the spindle is given by $\xi = \xi_0 < \pi$ and the exterior region is defined by $0 < \xi < \xi_0$. We may choose as particular solutions of Eq. (1) functions determined with the aid of (5) which are of the form

$$[(s - t)/(1 - t^2)^{1/2}]^{q+1/2} Q^{im}_{q-1/2}(i\lambda) \cos m\eta \tag{10}$$

where $2q = n - 3$, $s = \cosh \eta$, $t = \cos \xi$ and $\lambda = \cot \xi$. The Q function represents a generalized spherical harmonic of the second kind as defined by E. Hobson [7]. The functions considered in (10) obviously vanish at infinity ($\xi = 0$, $\eta = 0$). We are primarily interested in odd integral values of n and in this case (10) is considerably simplified. Particular solutions of (1) are then given by

$$(s - t)^{q+1/2} K^{(q)}_m(t) \cos m\eta \tag{11}$$

where (q) denotes the qth partial derivative with respect to the argument and $K_\alpha(t)$ is a Legendre function of complex degree commonly called a conal function [7, p. 445]. It is defined as

$$K_\alpha(t) = \left(\frac{2}{\pi}\right) \cosh \alpha\pi \int_0^\infty [2 \cosh u + 2 \cos \xi]^{-1/2} \cos \alpha u \, du. \tag{12}$$

If we replace ξ by $(\pi - \xi)$ in (12) we obtain

$$K_\alpha(-t) = \left(\frac{2}{\pi}\right) \cosh \alpha\pi \int_0^\infty [2 \cosh u - 2 \cos \xi]^{-1/2} \cos \alpha u \, du. \tag{13}$$

Now if $0 < \xi < \pi$, Eq. (13) may be differentiated q times with respect to $\cos \xi$ the order of differentiation and integration being interchanged on the right. For $0 < \xi_0 < \pi$ the term $(\cosh \eta - \cos \xi_0)^{-(q+1/2)}$ satisfies the conditions of the Fourier integral theorem and may be expanded as follows

$$(\cosh \eta - \cos \xi_0)^{-(q+1/2)} = \left(\frac{2}{\pi}\right) \int_0^\infty \cos \alpha\eta \left\{ \int_0^\infty [\cosh \eta' - \cos \xi_0]^{-(q+1/2)} \cos \alpha\eta' \, d\eta' \right\} d\alpha. \tag{14}$$

We notice that the term in braces is simply $K^{(q)}_\alpha(-t)$ multiplied by a function of α. We have used t_0 to represent $\cos \xi_0$. We understand by the superscript (q) in $K^{(q)}_\alpha(-t)$ the qth partial derivative with respect to $\cos \xi$ and not with respect to $-\cos \xi$.

We now choose for the electrostatic potential $\varphi\{n\}$ a function of the form

$$\varphi\{n\}(s - t)^{-(q+1/2)} = \int_0^\infty A(\alpha) K^{(q)}_\alpha(t) \cos \alpha\eta \, d\alpha. \tag{15}$$

The condition that $\varphi = 1$ for $\xi = \xi_0$ determines with the aid of (14) the function $A(\alpha)$ since (15) must be satisfied identically in η. The n-dimensional electrostatic potential $\varphi\{n\}$ is found for odd n to be

$$\varphi\{n\} = (2\pi)^{1/2}\Gamma^{-1}(q + 1/2)(s - t)^{q+1/2} \int_0^\infty \frac{K_\alpha^{(q)}(-t_0)K_\alpha^{(q)}(t) \cos \alpha\eta}{K_\alpha^{(q)}(t_0) \cosh \alpha\pi} \, d\alpha. \tag{16}$$

This integral converges for all η and for $0 \leqq \xi < \xi_0$.

If n is not an odd integer the electrostatic potential function becomes much more complicated in general. However, in the special case $n = 4$ the potential may be easily determined with the aid of (5) as

$$\varphi\{4\} = 2(s - t)(1 - t^2)^{-1/2} \int_0^\infty \frac{\sinh \alpha(\pi - \xi_0) \sinh \alpha\xi \cos \alpha\eta}{\sinh \alpha\pi \sinh \alpha\xi_0} \, d\alpha. \tag{17}$$

For odd values of n the stream function $\Psi\{n\}$ representing the flow about a spindle may now be obtained from (16) with the aid of (3). We have

$$\Psi\{n\} = \frac{U(c \sin \xi)^{2(q+1)}}{2(q + 1)(s - t)^{2(q+1)}} \cdot \left[1 - \frac{(2\pi)^{1/2}(s - t)^{q+3/2}}{\Gamma(q + 3/2)} \int_0^\infty \frac{K_\alpha^{(q+1)}(-t_0)K_\alpha^{(q+1)}(t) \cos \alpha\eta}{K_\alpha^{(q+1)}(t_0) \cosh \alpha\pi} \, d\alpha\right]. \tag{18}$$

Thus when $n = 3$ $(q = 0)$ we obtain the stream function corresponding to a uniform flow about a three-dimensional spindle. It is given by

$$\Psi\{3\} = \frac{Uc^2 \sin^2 \xi}{2(s - t)^2}\left[1 - (2)^{3/2}(s - t)^{3/2} \int_0^\infty \frac{K_\alpha^{(1)}(-t_0)K_\alpha^{(1)}(t) \cos \alpha\eta}{K_\alpha^{(1)}(t_0) \cosh \alpha\pi} \, d\alpha\right]. \tag{19}$$

If $\xi_0 = \pi/2$ the spindle becomes a sphere and the well known stream function for the sphere is obtained.

The capacity $C\{n\}$ of an n-dimensional spindle is found according to (8) as

$$C\{n\} = 2^{q+1}(2q + 1)\pi^{1/2}c^{2q+1}\Gamma^{-1}(q + 1/2) \int_0^\infty \frac{K_\alpha^{(q)}(-t_0)K_\alpha^{(q)}(1)}{K_\alpha^{(q)}(t_0) \cosh \alpha\pi} \, d\alpha. \tag{20}$$

The case $n = 3$ has been given by G. Szegö [8]. We obtain the virtual mass $M\{3\}$ with the aid of (7).

$$M\{3\} = -\left(\frac{2}{3}\right)\pi c^3\left\{2 + 3 \cot^2 \xi_0 + 3\xi_0 \cot \xi_0 \csc^2 \xi_0 + 3 \int_0^\infty \frac{(4\alpha^2 + 1)K_\alpha^{(1)}(-t_0)}{K_\alpha^{(1)}(t_0) \cosh \alpha\pi} \, d\alpha\right\}. \tag{21}$$

It is easily verified that for $\xi_0 = \pi/2$ we have the well known virtual mass of the sphere.

4. Flow About a Lens. Let us introduce the peripolar transformation, $x + iy = -c \cot \frac{1}{2}(\xi + i\eta)$, where c is a positive constant. The profile of a lens is defined by two curves $\xi = \xi_1$, and $\xi = \xi_2$. We shall assume that $0 < \xi_1 < \xi_2 < 2\pi$. The external region is chosen as $\eta > 0$, $\xi_2 - 2\pi < \xi < \xi_1$. Particular solutions of (1) are given by functions of the type

$$(s - t)^{q+1/2}(s^2 - 1)^{-q/2}K_m^q(s) \cosh m\xi \tag{22}$$

and those obtained by replacing $\cosh m\xi$ by $\sinh m\xi$. In (22) $K_m^q(s)$ is a Legendre function of the type discussed by Mehler, see [7, p. 451]. As in the case of the spindle the lens problem is solved much more easily and the solution is given in a much simpler form whenever n is an odd integer. Particular solutions of (1) are in this case given by

$$(s - t)^{q+1/2} K_m^{(q)}(s) \cosh m\xi. \tag{23}$$

solutions containing $\sinh m\xi$ being understood as before. The function $K_\alpha(s)$ is defined [7, p. 451] as

$$K_\alpha(s) = \left(\frac{2}{\pi}\right) \cosh \alpha\pi \int_0^\infty [2 \cosh u + 2 \cosh \eta]^{-1/2} \cos \alpha u \, du. \tag{24}$$

We have also the known expansion [7, pp. 452, 453] valid for $0 < \xi < 2\pi$

$$(s - t)^{-1/2} = 2^{1/2} \int_0^\infty \cosh \alpha(\xi - \pi) \operatorname{sech} \alpha\pi \, K_\alpha(s) \, d\alpha. \tag{25}$$

Clearly $K_\alpha^{(q)}(s)$ is a well defined function obtained from (24) by a permissible exchange of order of integration and differentiation. Also for $0 < \xi < 2\pi$ Eq. (25) may be differentiated q times with respect to s the order of integration and differentiation being interchanged on the right. We choose as the electrostatic potential $\varphi\{n\}$ of the lens

$$\varphi\{n\}(s - t)^{-(q+1/2)} = \int_0^\infty [A(\alpha) \cosh \alpha\xi + B(\alpha) \sinh \alpha\xi] K_\alpha^{(q)}(s) \, d\alpha. \tag{26}$$

The functions $A(\alpha)$ and $B(\alpha)$ may be chosen in such a way that $\varphi\{n\} = 1$ for $\xi = \xi_1$ and $\xi = \xi_2$. We need only differentiate (25) q times with respect to s evaluate for $\xi = \xi_1$ and $\xi = \xi_2$ and insert in (26). Since (26) must be satisfied identically in η the functions $A(\alpha)$ and $B(\alpha)$ are easily determined. The electrostatic potential of an n-dimensional lens (n odd) is thus given by

$$\varphi\{n\} = (-1)^q (2\pi)^{1/2} \Gamma^{-1}(q + 1/2)(s - t)^{q+1/2} \int_0^\infty F(\alpha, \xi) \operatorname{sech} \alpha\pi \, K_\alpha^{(q)}(s) \, d\alpha \tag{27}$$

where

$$\sinh \alpha(2\pi - \xi_2 + \xi_1) F(\alpha, \xi)$$
$$= \sinh \alpha(\xi_1 - \xi) \cosh \alpha(\pi - \xi_2) + \cosh \alpha(\pi - \xi_1) \sinh \alpha(2\pi - \xi_2 + \xi). \tag{28}$$

Equation (27) is valid for all positive η and for all ξ in the interval $\xi_2 - 2\pi < \xi < \xi_1$. The case $n = 3$ was given by F. G. Mehler [5] in 1868.

We note again that in case $n = 4$ the electrostatic potential may be easily obtained with the aid of (5). We have

$$\varphi\{4\} = 2[(s - t)/(s^2 - 1)^{1/2}] \int_0^\infty F(\alpha, \xi) \operatorname{csch} \alpha\pi \sin \alpha\eta \, d\alpha. \tag{29}$$

The stream function $\Psi\{n\}$ representing the flow about an odd-dimensional lens is obtained from (27) with the aid of (3). Thus for $n = 3$

$$\Psi\{3\} = [Uc^2(s^2 - 1)/2(s - t)^2] \left[1 + 2^{3/2}(s - t)^{3/2} \int_0^\infty F(\alpha, \xi) \operatorname{sech} \alpha\pi \, K_\alpha^{(1)}(s) \, d\alpha\right]. \tag{30}$$

By an appropriate choice of ξ_1 and ξ_2 electrostatic potentials and stream functions for a spherical bowl, symmetrical lens, hemisphere, *etc.*, may be determined.

Because of the invariance of form of the Stokes-Beltrami equations, i.e.

$$y^{n-2}\Phi_\xi = \Psi_\eta\,, \qquad y^{n-2}\Phi_\eta = -\Psi_\xi \tag{31}$$

it is a simple matter to determine from Ψ the velocity potential Φ to which Ψ is associated. It remains to be shown that this soluion is unique.

The problem of establishing the uniqueness of the stream function Ψ defined in the infinite region outside the profile in the xy plane and satisfying prescribed boundary conditions is equivalent to the problem of establishing uniqueness of this function in an infinite half strip in the $\xi\eta$ plane. An application of Green's formula would demand a knowledge of the behavior of the derivatives of Ψ at infinity in the $\xi\eta$ plane. Hence we find it more convenient to make use of an eigen value method due to A. Weinstein [9] which requires only a knowledge of the behavior of Ψ at infinity. By this method we can show that there is only one stream function Ψ which satisfies the prescribed conditions on the lens profile. If the stream function is unique the potential Φ is also unique up to an arbitrary constant which must be zero in order that the potential vanish at infinity.

The electrostatic capacity of an n-dimensional lens is determined for integral values of q (n odd) with the aid of Eq. (8). It is given by

$$C\{n\} = \frac{(-1)^q 2^{q+1}\Gamma(1/2)(2q+1)c^{2q+1}}{\Gamma(q+1/2)}$$

$$\cdot \int_0^\infty \frac{\sinh \alpha\xi_1 \cosh \alpha(\pi - \xi_2) + \cosh \alpha(\pi - \xi_1) \sinh \alpha(2\pi - \xi_2)}{\sinh \alpha(2\pi + \xi_1 - \xi_2) \cosh \alpha\pi} K_\alpha^{(q)}(1)\, d\alpha. \tag{32}$$

The case $n = 3$ has been given by G. Szego [8]. The virtual mass $M\{3\}$ is obtained with the aid of [7] and given by

$$M\{3\} = 2\pi c^3 \int_0^\infty \frac{\sinh \alpha\xi_1 \cosh \alpha(\pi - \xi_2) + \cosh \alpha(\pi - \xi_1) \sinh \alpha(2\pi - \xi_2)}{\sinh \alpha(2\pi + \xi_1 - \xi_2) \cosh \alpha\pi}$$

$$\cdot (4\alpha^2 + 1)\, d\alpha - V\{3\} \tag{33}$$

where

$$V\{3\} = (\pi c^3/6)\{(2 - \cos \xi_1) \cot \xi_1/2 \csc^2 \xi_1/2 - (2 - \cos \xi_2) \cot \xi_2/2 \csc^2 \xi_2/2\}.$$

Equation (33) is a much simpler expression for the virtual mass than that given by Shiffman and Spencer. Several special cases may be obtained easily from (33). In particular the virtual mass of a hemisphere is given by

$$M\{3\} = (2\pi c^3/81)[135 - 59(3)^{1/2}] = 2.545c^3 \tag{34}$$

where c is the radius of the hemisphere.

5. Additional Results. We shall list here the electrostatic potentials of certain other n-dimensional bodies of revolution. With the aid of (3), (7) and (8) the corresponding flow problems can be completely solved. The results given in this section are valid for any positive real value of $q(n > 2)$. It will be noted that the results are simplified considerably whenever n is an odd integer.

I. *Sphere*: The potential of an n-dimensional sphere of radius a is given as

$$\varphi\{n\} = (a/r)^{n-2}, \qquad r = (x^2 + y^2)^{1/2}. \tag{35}$$

II. *Two separated spheres*: The lines $\eta = \alpha < 0$ and $\eta = \beta > 0$ in dipolar coordinates define two separated spheres. The potential in this case is given by

$$\varphi\{n\} = 2^{q+1/2}(s - t)^{q+1/2} \sum_{n=0}^{\infty} \frac{e^{-N\beta} \sinh N(\eta - \alpha) + e^{N\alpha} \sinh N(\beta - \eta)}{\sinh N(\beta - \alpha)} P_n(t/2q + 1) \tag{36}$$

where $N = n + q + \frac{1}{2}$ and $P(t \mid 2q + 1)$ represents the $2q + 1$-dimensional zonal spherical harmonic or more commonly the Gegenbauer polynomial.

III. *Prolate Spheroid*: A line $\xi = \xi_0$ defines a prolate spheroid under the transformation $z = c \cosh \zeta$. The electrostatic potential for such a spheroid is

$$\varphi\{n\} = (\rho_0/\rho)^q Q_q^q(s)/Q_q^q(s_0) \tag{37}$$

where $\rho = \sinh \xi$, $s = \cosh \xi$.

IV. *Oblate Spheroid*: Under the transformation $z = c \sinh \zeta$ an oblate spheroid is defined by a line $\xi = \xi_0$, and the potential is given as:

$$\varphi\{n\} = (s_0/s)^q Q_q^q(i\rho)/Q_q^q(i\rho_0). \tag{38}$$

V. *Disc*: If $\xi_0 = 0$ the oblate spheroid becomes a disc of radius c and the potential of such a disc is obtained from (38) as

$$\varphi\{n\} = -\left[2^{q-1} \exp\left\{\left(\frac{q+1}{2}\right)\pi i\right\}\Gamma\left(q + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)s^q\right]^{-1} Q_q^q(i\rho). \tag{39}$$

We have listed here only a few examples. Numerous others can be easily obtained.

6. Internal Problems. The method of Generalized Electrostatics is also useful in determining the flow induced in a fluid contained between two or more axially symmetric boundaries when one or more of the boundaries moves with respect to the others at uniform velocity parallel to the axis of symmetry. In this case we consider instead of Eq. (3) the equation

$$\Psi\{n\} = Uy^{n-1}(n - 1)^{-1}\varphi\{n + 2\}. \tag{40}$$

On a moving boundary $\Psi\{n\} = V_i y^{n-1}(n - 1)^{-1}$ where $V_i = c_i U$ (c_i is a constant possibly differing for each moving boundary). On a stationary boundary $\Psi\{n\} = 0$. This problem is reduced by (40) to the solution of a steady state heat flow problem in $n + 2$-dimensions. The boundaries in $n + 2$-space corresponding to the moving boundaries in n-space are maintained at temperatures c_i and those corresponding to the stationary boundaries are kept at temperature 0. This procedure applies in particular to the case in which the fluid is bounded by two eccentric spheres, two tori or to the case in which one portion of the boundary moves with respect to another portion in an infinite fluid.

7. Concluding Remarks. In this paper we have considered only three dimensional flow problems. It should be remarked, however, that a similar procedure may be employed in solving plane flow problems for profiles symmetric about the x-axis. In fact if the profile possesses symmetry with respect to both axes the plane problem may be solved for uniform flow in any direction.

Bibliography

1. A. Weinstein, *The method of singularities in the physical and in the hodograph plane,* Fourth Symp. Appl. Math. (in print); see also, *Discontinuous integrals and generalized axially symmetric potential theory,* Trans. Amer. Math. Soc. **63**, 342–354 (1948).
2. L. E. Payne and A. Weinstein, *Capacity, virtual mass and symmetrization* (in print).
3. A. Weinstein, *On cracks and dislocations in shafts under torsion,* Q. Appl. Math. (in print).
4. M. Shiffman and D. C. Spencer, *The flow of an ideal incompressible fluid about a lens,* Q. Appl. Math. **5**, 270–288 (1947).
5. F. G. Mehler, *Ueber die Vertheilung der statischen Elektricität in einem von zwei Kugelkalotten begrenzten Körper,* J. R. Angew. Math. **68**, 134–150 (1868).
6. C. Snow, *The hypergeometric and Legendre functions with applications to integral equations and potential theory.* National Bureau of Standards, Washington, D. C. (1942).
7. E. Hobson, *Spherical and ellipsoidal harmonics,* Cambridge University Press (1931).
8. G. Szegö, *On the capacity of a condenser,* Bull. Amer. Math. Soc., **51**, 325–350 (1945).
9. A. Weinstein, *On surface waves,* Can. J. Math. (3) **1**, 271–278 (1949).
10. M. Schiffer and G. Szegö, *Virtual mass and polarization,* Trans. Amer. Math. Soc. (1) **67**, 130–205 (1949).

A FORMULA FOR AN INTEGRAL OCCURRING IN THE THEORY OF LINEAR SERVOMECHANISMS AND CONTROL-SYSTEMS*

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Introduction. Let t denote the time, $p = d/dt$ the differential operator with respect to time and

$$f_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n ; \qquad a_0 \neq 0, n \geq 1 \tag{1}$$

a polynomial with real coefficients. If all zeros of $f_n(x)$ have negative real parts, every solution $y(t)$ of

$$f_n(p)y = 0 \tag{2}$$

and all derivatives p^ky tend to zero with increasing t. Moreover the integral

$$Y = \int_0^\infty y^2(t)\, dt \tag{3}$$

exists. The purpose of this paper is to develop a formula for Y in terms of squared linear forms of the initial values

$$p^ky(0) = q_k ; \qquad k = 0, 1, \cdots, n - 1. \tag{4}$$

No further quantities but the coefficients a_i of (1) shall appear in this formula.

Such a formula may be useful for the design of linear servomechanisms and control-systems, governed by the equation

$$f_n(p)y = z(t). \tag{2'}$$

where $z(t)$ may be considered as an arbitrary disturbance function. For instance, let $z(t) \equiv 1$ for $t < 0$. At $t = 0$, $z(t)$ may step down to $z(t) \equiv 0$ for $t \geq 0$. The response $y(t)$ then is a solution of (2), and the integral Y measures, how fast the systems lines up with the stepping of z. The knowledge of Y makes it possible to choose the coefficients a_i of (1) under given conditions in order to minimize Y**. Two examples of such a minimization will be given in Sec. 4.

The development of this formula will also yield a new approach to the well known Hurwitz criterion of stability and to reductions of "stable" operator polynomials in p to such of a lower degree, including the reduction of Schur [1].

1. Auxiliary theorems and algorithms of reduction. Notation. Let J be the imaginary axis of the complex plane, J' the set of all points ωi, of J with $\omega > 0$, J'' the set of all points ωi of J with $\omega < 0$, and Re x the real, Im x the imaginary part of x.

Definitions. Let $f(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m$ a polynomial with real or complex coefficients. We call m the proper degree of $f(x)$, if $b_0 \neq 0$. We define now
1) $F(x) = b_0x^m + b_m$ as the "simplification" of $f(x)$, if $f(x)$ has the proper degree m,

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**Minimization of Y has already been investigated by P. Hazebroek and B. L. van der Waerden [2] who also gave a formula expressing Y as a symmetric function of the zeros of (1) for special systems (4).

2) $g(x) = b_m + b_{m-2}x^2 + \cdots$ as the even and $h(x) = f(x) - g(x)$ as the odd component of $f(x)$,
3) $f(x)$ as a "Hurwitz-polynomial", if all zeros of $f(x)$ are in the left-hand half-plane $\operatorname{Re} x < 0$ (the case $f(x) \equiv$ const. $\neq 0$ to be included),
4) $f(x)$ as definite (semidefinite) on a given set M of points of the complex plane, if a suitable constant $c \neq 0$ can be found, so that $cf(x) > 0$ (≥ 0) on M (for instance, x^m is definite on J'), c to be normalized by $|c| = 1$.

Lemma 1. Let $p(x)$ and $q(x)$ be two polynomials. The linear combination $r(x, t) = tp(x) + (1 - t)q(x)$ shall have proper degree m for all values $0 \leq t \leq 1$. We further assume $r(x, t) \neq 0$ on J for all these values of t. Then $p(x)$ and $q(x)$ have the same number of zeros for $\operatorname{Re} x > 0$ and for $\operatorname{Re} x < 0$.

Proof. No zero of $r(x, t)$ can pass J or can go to infinity, when t is running from 0 to 1. Hence the number of zeros for $\operatorname{Re} x > 0$ remains constant. The same holds for $\operatorname{Re} x < 0$.

Lemma 2. Let $f(x) = b_0x^m + \cdots + b_m$ be a polynomial with real coefficients b_k. The proper degree shall be $m \geq 1$; $f(x)$ and its simplification $F(x)$ shall not vanish on J. Then $f(x)$ and $F(x)$ have the same number of zeros for $\operatorname{Re} x > 0$ and for $\operatorname{Re} x < 0$, if at least one of the following conditions is satisfied:
a) the even component $g(x)$ of $f(x)$ is semidefinite on J';
b) m is odd, and the odd component $h(x)$ of $f(x)$ is semidefinite on J';
c) m is even, and the odd component $h(x)$ of $f(x)$ is definite on J'.

Proof. We set $r(x, t) = tf(x) + (1 - t)F(x) = b_0x^m + tb_1x^{m-1} + \cdots + tb_{m-1}x + b_m$. The proper degree of $r(x, t)$ is m for all values of t. We shall prove that $r(x, t) \neq 0$ on J for $0 \leq t \leq 1$. Application of the first lemma then completes the proof.

From the assumptions it follows that $r(x, 0) \neq 0$ on J and $r(x, 1) \neq 0$ on J; furthermore $r(0, t) = b_m \neq 0$. It is therefore sufficient to prove $r(x, t) \neq 0$ on J' or J'' for $0 < t < 1$. We denote by $G(x)$ the even and by $H(x)$ the odd component of $F(x)$. $G(x)$ is either the simplification of $g(x)$ or equal to $g(0) = b_m$; $H(x)$ is either the simplification of $h(x)$ or equal to $h(0) = 0$.

If any polynomial $s(x)$ is semidefinite on J', we have $cs(x) \geq 0$ on J' with a suitable constant c ($|c| = 1$). Considering extremely small and extremely great values of $|x|$, we find $cS(x) \geq 0$ on J' with the same constant c for the simplification $S(x)$ of $s(x)$. With this in mind, we distinguish the following three cases according to the conditions a, b, c of the lemma.

a) $g(x)$ is semidefinite on J'. This leads to $cg(x) \geq 0$ and to $cG(x) \geq 0$ on J'. We have either $G(x) = b_m$ or $G(x) = F(x)$, and in both cases $G(x) \neq 0$ on J. Therefore,

$$|r(x, t)| \geq |\operatorname{Re} r(x, t)| = c.tg(x) + c.(1 - t)G(x) \geq (1 - t)cG(x) > 0 \text{ on } J'.$$

b) m odd, $h(x)$ semidefinite on J'. We have $H(x) = b_0x^m \neq 0$ on J' and a suitable constant c, making $ch(x) \geq 0$ and $cH(x) \geq 0$ on J'. Hence for $0 < t < 1$ on J'

$$|r(x, t)| \geq |\operatorname{Im} r(x, t)| = c.th(x) + c.(1 - t)H(x) \geq c(1 - t)H(x) > 0.$$

c) m even, $h(x)$ definite on J'. We find $|r(x, t)| \geq t\,|h(x)| > 0$ on J' for $0 < t < 1$. Thus, $r(x, t) \neq 0$ on J.

Lemma 3. Let $p(x)$ and $q(x)$ be any two polynomials with real coefficients, p having proper degree m and q having proper degree $m' < m$. The polynomial $f(x) = p(x)q(-x)$ and its simplification $F(x)$ shall not vanish on J; $f(x)$ and $F(x)$ shall have the same number of zeros for $\operatorname{Re} x > 0$ and for $\operatorname{Re} x < 0$. From this it follows that

a) if $p(x)$ is a Hurwitz-polynomial, $q(x)$ is also one with $m' = m - 1$,
b) if $q(x)$ is a Hurwitz-polynomial, if furthermore $m = m' + 1$, and if all coefficients of $p(x)$ are positive, then $p(x)$ is also a Hurwitz-polynomial.

Proof. The number of zeros of $F(x)$ in the half-plane Re $x < 0$ may be n, the number of zeros in Re $x > 0$ may be n'. All zeros of $F(x)$ form a regular polygon for $n + n' \geq 3$, and no zero can appear on J. Hence $|n - n'| \leq 1$. Should $p(x)$ be a Hurwitz polynomial, $f(x)$ and $F(x)$ have at least m zeros in Re $x < 0$ and not more than $m' \leq m - 1$ zeros in Re $x > 0$. Therefore $n = m$ and $n' = m' = m - 1$. The $m - 1$ zeros of $f(x)$ in Re $x > 0$ are those of $q(-x)$. This means, that $q(x)$ is a Hurwitz polynomial. Should the conditions of b) be satisfied, then at least $m - 1$ zeros of $F(x)$ and consequently of $f(x)$ appear in Re $x < 0$. Therefore $p(x)$ has $m - 1$ zeros in Re $x < 0$. Should the last zero of $p(x)$ be situated in Re $x > 0$, it must necessarily be real, i.e. positive. But no such zero can exist, since $p(x)$ is assumed to have positive coefficients. This completes the proof of the lemma.

Note. The condition, that all coefficients of $p(x)$ are positive is—apart from a constant factor—a necessary condition for $p(x)$ to be a Hurwitz polynomial. It is well known and it can easily be proved by splitting $p(x)$ into root factors. No coefficient can vanish without reducing the degree of the polynomial.

Algorithms can be based on Lemmas 2 and 3 in order to reduce a Hurwitz polynomial to such of a lower degree. It may be worthwhile to explain, how the well-known reduction of Schur (see [1]) can be obtained in this way.

Schur's algorithm of reduction. We consider the polynomial (1) with real coefficients, but we do not assume that it is a Hurwitz polynomial. We denote by g^+ the even and by h^+ the odd component of $f_n(x)$. With Schur we introduce

$$f_{n-1}(x) = (2a_1 - a_0x)[g^+(x) + h^+(x)] + (-1)^n a_0 x[g^+(x) - h^+(x)] \tag{5}$$

with lower degree than n. The odd component of the polynomial $f(x) = f_n(x)f_{n-1}(-x)$ is

$$h(x) = -2a_0xh^{+2}(x) \text{ for even } n, \qquad h(x) = 2a_0xg^{+2}(x) \text{ for odd } n. \tag{6}$$

This component is obviously semidefinite on J' and on J''. It can easily be seen, that $f_n(x) = 0$ on J in any point x leads to $f_{n-1}(x) = 0$ for the same point. *Vice versa*, $a_1 f_n(x) = 0$ is a consequence of $f_{n-1}(x) = 0$ in any point x of J. We now assume that

$$a_1 \neq 0. \tag{7}$$

This is necessary and sufficient for f_{n-1} to have the proper degree $n - 1$. The polynomial $f(x)$ then has the proper degree $2n - 1$. If either f_n or f_{n-1} is a Hurwitz polynomial, f cannot vanish on J. Also $F(x)$, the simplification of f, cannot vanish on J. Hence Lemma 2 is applicable to f and F and the Lemma 3 to f_n and f_{n-1}. Thus, if f_n is a Hurwitz polynomial, f_{n-1} is also one; if f_{n-1} is a Hurwitz polynomial and if f_n has positive coefficients, then f_n is a Hurwitz polynomial too.

Another algorithm. Assume that $f_n(x)$ and $f_n(-x)$ do not have common zeros. Then two polynomials $r(x)$ and $t(x)$ with real coefficients and with no higher degree than $n - 1$ exist, satisfying

$$f_n(x)r(x) + f_n(-x)t(x) \equiv 2. \tag{8}$$

From this it follows that

$$f_n(x)f_{n-1}(-x) + f_n(-x)f_{n-1}(x) \equiv 2 \qquad \text{with} \qquad 2f_{n-1}(x) = r(-x) + t(x), \tag{9}$$

the degree of f_{n-1} being at most $n - 1$. Any other polynomial $p(x)$ of any degree, which satisfies (9) instead of f_{n-1} can be written as

$$p(x) = f_{n-1}(x) + s(x)f_n(x), \tag{10}$$

with a suitable odd polynomial $s(x) = -s(-x)$. Hence f_{n-1} is the only polynomial satisfying (9) with no higher degree than $n - 1$. From (9) it follows that

$$g(x) \equiv 1 \tag{11}$$

for the even component of the product $f(x) = f_n(x)f_{n-1}(-x)$; $g(x)$ is definite on J' and on J'', it is even definite on J. The product $f(x)$ cannot vanish on J. Also its simplification $F(x)$ cannot vanish on J, since the degree of $F(x)$ is odd and $F(0) = 1$. Therefore $f(x)$ and $F(x)$ have the same number of zeros for $\operatorname{Re} x > 0$ and for $\operatorname{Re} x < 0$ according to Lemma 2. Lemma 3 is applicable to f_n and f_{n-1}. Thus, if f_n is a Hurwitz polynomial, f_{n-1} is also one with proper degree $n - 1$. If f_{n-1} with proper degree $n - 1$ is a Hurwitz polynomial and if f_n has positive coefficients, f_n is a Hurwitz polynomial too, and this is a consequence of (9).

2. Details concerning the second algorithm of reduction. The second algorithm will be useful for the development of the formula announced. Some necessary details will therefore be developed. We assume $f_n(x) = a_0x^n + \cdots + a_n$ to be a Hurwitz polynomial of proper degree n with real coefficients. As already stated, the polynomial $f_{n-1}(x)$ defined by (9) is also a Hurwitz polynomial with real coefficients and with proper degree $n - 1$. The method leading from f_n to f_{n-1} can now be applied to f_{n-1} and so on. Thus we obtain a sequence of Hurwitz polynomials

$$f_n, f_{n-1}, f_{n-2}, \cdots, f_1, f_0, \tag{12}$$

with f_0 as a constant; f_k has the proper degree k and real coefficients; any two adjacent polynomials f_k, f_{k-1} satisfy

$$f_k(x)f_{k-1}(-x) + f_k(-x)f_{k-1}(x) \equiv 2. \tag{13}$$

It means no loss of generality to assume

$$f_n(0) = a_n = 1; \tag{14}$$

(13) and (14) then lead to

$$f_k(0) = 1 \qquad \text{for} \qquad k = 0, 1, \cdots, n = 1. \tag{15}$$

This in turn causes positive coefficients for all polynomials f_k (see Sec. 1, Note). We increase all subscripts in (13) by 1 and subtract the new equation from (13); hence $p(x)f_k(-x) + p(-x)f_k(x) \equiv 0$ with $p(x) = f_{k+1}(x) - f_{k-1}(x)$; $f_k(x)$ and $f_k(-x)$ have no common zeros. Therefore,

$$f_{k+1}(x) - f_{k-1}(x) = c_{k+1}\, x\cdot f_k(x) \qquad \text{for} \qquad k = 1, 2, \cdots, n - 1 \tag{16}$$

with a suitable constant

$$c_{k+1} > 0. \tag{16'}$$

In addition to (16), we write

$$f_1(x) = 1 + c_1x; \qquad c_1 > 0. \tag{16''}$$

Regarding the positive constants $c_1, c_2, \cdots, c_n$ as given, we can solve the system (16) with regard to $f_2, \cdots, f_n$. We find:

$$f_k(x) = \begin{vmatrix} 1 + c_1x & -1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 1 & c_2x & -1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & c_3x & -1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 1 & c_kx \end{vmatrix}; \qquad k = 2, 3 \tag{17}$$

This is a representation of all Hurwitz polynomials of proper degree k with $f_k(0) = 1$. Vice versa all determinants (17) with coefficients $c_i > 0$ give Hurwitz polynomials. Another representation may be given by means of the determinants

$$\begin{vmatrix} c_1x & -1 & 0 & \cdot & \cdot & 0 & 0 \\ 1 & c_2x & -1 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & c_3x & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 & c_kx \end{vmatrix} = F(x; c_1, c_2, \cdots c_k). \tag{18}$$

We can write then

$$f_k(x) = F(x; c_1, c_2, \cdots, c_k) + F(x; c_2, c_3, \cdots, c_k), \tag{17'}$$

the right-hand-side showing the even and the odd component of f_k. The functions (18) have imaginary zeros in x or real zeros in ix, which can easily be recognized as the eigenvalues of a Hermitian matrix. The result about the zeros of the components of a Hurwitz polynomial with real coefficients is well known and has been found by E. J. Routh. So far this represents a minor application of (18).

We are now going to develop another formula for f_k where only the coefficients a_i of the given polynomial f_n appear. For this purpose we introduce the column-vector

$$\mathfrak{a}_{ij} = \begin{pmatrix} a_{i-2j+2} \\ a_{i-2j+4} \\ \cdot \\ \cdot \\ \cdot \\ a_i \end{pmatrix}, \qquad a_k = 0 \quad \text{for} \quad k > n \quad \text{and for} \quad k < 0;$$

with j components, the matrices

$$\mathfrak{H}_k = (\mathfrak{a}_{2k-1,k}, \mathfrak{a}_{2k-2,k}, \cdots \mathfrak{a}_{k,k}); \qquad k = 1, 2, \cdots, n,$$

the so-called Hurwitz determinants

$$D_0 = \operatorname{sign} a_0 = 1, \qquad D_k = \| \mathfrak{H}_k \| \quad \text{for} \quad k = 1, 2, \cdots, n, \tag{19}$$

and the column-vector

$$\beta_k = \begin{pmatrix} b_{0k} \\ \cdot \\ \cdot \\ b_{k-1,k} \end{pmatrix}$$

of the coefficients of the polynomial

$$f_k(-x) = b_{0k}x^k + b_{1k}x^{k-1} + \cdots + b_{k-1,k}x + 1.$$

We then consider the polynomials

$$f_n(x)f_k(-x) - (-1)^{n-k}f_n(-x)f_k(x) = w_{n-k-1}(x); \qquad k = 0, 1, \cdots, n \tag{20}$$

with the two significant special cases

$$w_{-1}(x) \equiv 0, \qquad w_0(x) \equiv 2. \tag{20'}$$

From (16) and (20) follows for $n - k \geq 1$

$$w_{n-k} = c_{k+1}xw_{n-k-1} + w_{n-k-2}, \tag{21}$$

and we derive from (16'), (20') and (21) that $w_{n-k}(x)$ has the proper degree $n - k$. This means: the product $f_n(x)f_k(-x)$ does not contain the powers $x^{n+k-1}, x^{n+k-3}, \cdots, x^{n-k+1}$. This is expressed by

$$\mathfrak{H}_k\beta_k = -\mathfrak{a}_{k-1,k}. \tag{22}$$

There is only one polynomial f_{n-1} of degree $n - 1$ according to (9). Hence there is only one solution β_{n-1} of (22) for $k = n - 1$, and this leads to $D_{n-1} \neq 0$. Let $D_{k+1} \neq 0$; consequently the matrix $\mathfrak{H}_{k+1}$ is of rank $k + 1$, while the matrix $(\mathfrak{H}_k, \mathfrak{a}_{k-1,k})$ consisting of all rows but the last of $\mathfrak{H}_{k+1}$ is of rank k. This very matrix appears in (22), so only one solution of (22) for β_k exists, and therefore $D_k \neq 0$. Hence

$$D_k \neq 0; \qquad k = 1, 2, \cdots, n - 1. \tag{23}$$

All systems (22) have only one solution β_k, and this belongs to

$$f_k(x) = \frac{1}{D_k} \begin{vmatrix} x^k & -x^{k-1} & \cdot\cdot & (-1)^k \\ \mathfrak{a}_{2k-1,k} & \mathfrak{a}_{2k-2,k} & \cdot\cdot & \mathfrak{a}_{k-1,k} \end{vmatrix} = a_0 \frac{D_{k-1}}{D_k} x^k + \cdots + 1. \tag{24}$$

The proof is clear. The coefficients of f_k are positive. We have $D_1 = a_1 > 0$, $a_0D_{k-1}D_k^{-1} > 0$, $D_n = a_nD_{n-1}$ and thus,

$$D_i > 0 \qquad \text{for} \qquad i = 1, 2, \cdots, n. \tag{25}$$

The coefficients c_i in (16) are the quotients of the highest-power-terms of f_i and f_{i-1}. Therefore

$$c_1 = a_0 a_1^{-1}, \qquad c_k = D_{k-1}^2 D_k^{-1} \cdot D_{k-2}^{-1} \quad \text{for} \quad k = 2, 3, \cdots, n. \tag{26}$$

The inequalities (25) form the well known Hurwitz criterion of stability.

3. The formula for Y. Let

$$P(u, v) = \sum_{i,k=0}^{m} a_{ik} u^i v^k \tag{27}$$

be a polynomial of two variables u and v. Let $y(t)$ and $z(t)$ be two functions with continuous derivatives $p^i y$, $p^k z$ up to the order $i, k = m + 1$. We then set

$$P^*(y, z) = \sum_{i,k=0}^{m} a_{ik} p^i y \cdot p^k z. \tag{28}$$

We introduce $Q(u, v) = (u + v)P(u, v)$. Obviously,

$$\int_a^b Q^*(y, z)\, dt = \int_a^b pP^*(y, z)\, dt = P^*(y, z) \Big|_a^b \tag{29}$$

We consider the special polynomials

$$Q_k(u, v) = f_k(u) f_{k-1}(v) + f_k(v) f_{k-1}(u) - 2; \qquad k = 1, 2, \cdots, n. \tag{30}$$

From (16) it follows that

$$Q_k(u, v) = (u + v) c_k f_{k-1}(u) f_{k-1}(v) + Q_{k-1}(u, v). \tag{31}$$

Therefore,

$$Q_n(u, v) = (u + v) \sum_{k=1}^{n} c_k f_{k-1}(u) f_{k-1}(v). \tag{32}$$

We apply (29) to (32) with y and z as solutions of (2), i.e., $f_n(p)y = 0$ and $f_n(p)z = 0$. Hence

$$2 \int_a^b y(t) z(t)\, dt = -\int_a^b Q_n^*(y, z)\, dt = \sum_{k=1}^{n} c_k f_{k-1}^*(y) f_{k-1}^*(z) \Big|_b^a \tag{33}$$

with $f_{k-1}^*(y) = f_{k-1}(p)y$ and $f_{k-1}^*(z) = f_{k-1}(p)z$. Setting $y = z$ and $a = 0$, $b = \infty$ we find the announced formula

$$2Y = 2 \int_0^\infty y^2(t)\, dt = \sum_{k=1}^{n} c_k (f_{k-1}^*(y)_0)^2. \tag{34}$$

We express c_k and f_{k-1} according to (26) and (24). We obtain

$$2Y = a_0 a_1^{-1} q_0^2 + \sum_{k=1}^{n-1} D_{k-1}^{-1} D_{k+1}^{-1} \begin{vmatrix} q_k & -q_{k-1} & \cdot\cdot & (-1)^k q_0 \\ \mathfrak{a}_{2k-1,k} & \mathfrak{a}_{2k-2,k} & \cdot\cdot & \mathfrak{a}_{k-1,k} \end{vmatrix}^2 \tag{35}$$

with the initial values q_k as explained by (4). In this formula, squared linear forms of the q_k appear together with the coefficients a_i of the given equation. Formula (35) has already a form which makes it independent of the restriction $a_n = 1$. It holds quite generally.

In the special case $q_0 = 1$, $q_1 = q_2 = \cdots = q_{n-1} = 0$ we find

$$2Y = \sum_{k=1}^{n} c_k \,. \tag{36}$$

This sum can be easily computed from (16). Addition of all formulae (16) gives

$$f_n + f_{n-1} - 2 = -2 + f_0 + f_1 + \sum_{i=1}^{n-1} (f_{i+1} - f_{i-1}) = x \sum_{i=0}^{n-1} c_{i+1} f_i$$

or $\sum_{i=1}^{n} c_i$ = coefficient of x in $(f_n + f_{n-1})$.
Therefore

$$2Y = a_{n-1}a_n^{-1} + D_n^{-1} \cdot | \, \mathfrak{a}_{2n-3,n-1}\mathfrak{a}_{2n-4,n-1} \cdots \mathfrak{a}_{n,n-1}\mathfrak{a}_{n-2,n-1} \, |. \tag{36'}$$

This formula too is not restricted to $a_n = 1$.

4. Two applications. 1) We set $a_0 = a_n = 1$, which is no essential restriction. All other coefficients of f_n may be variable in order to minimize Y according to (36). This means, that the sum of all coefficients c_i is to be minimized under the restriction $c_1c_2 \cdots c_n = 1$. An elementary calculation gives Min $2Y = n$ for $c_1 = c_2 = \cdots = c_n = 1$ with

$$\begin{aligned} f_n(x) = x^n &+ \binom{n-1}{1} x^{n-2} + \binom{n-2}{2} x^{n-4} + \cdots \\ &+ \binom{n-1}{0} x^{n-1} + \binom{n-2}{1} x^{n-3} + \cdots \,. \end{aligned} \tag{37}$$

This formula can be proved by induction on n.

(2) There are servomechanisms with an arbitrary input $\theta_i(t)$ and with a servo controlled output $\theta_0(t)$. The control depends on

$$\epsilon(t) = \theta_0(t) - \theta_i(t) \tag{38}$$

and shall make $|\epsilon|$ as small as possible. According to the definitions given in [2], ϵ can be called the regulated variable and θ_0 the regulating flow. Let the servocontrol be of the proportional plus integral type, i.e.

$$a_0\ddot{\theta}_0 + a_1\dot{\theta}_0 = -a_2\epsilon - \int_0^t a_3 \, \epsilon \, dt \tag{39}$$

with constants $a_i > 0$ for $i = 0, 1, 2, 3$. Combination of (38) and (39) gives

$$a_0 \dddot{\epsilon} + a_1\ddot{\epsilon} + a_2\dot{\epsilon} + a_3\epsilon = -a_0\dddot{\theta}_i - a_1\ddot{\theta}_i \,. \tag{40}$$

Due to the integral in (39), $\epsilon(t)$ tends to zero with increasing t if the right-hand-side vanishes identically and if

$$D_2 = a_1a_2 - a_0a_3 > 0. \tag{41}$$

Now we consider the case

$$\theta_i \equiv 0 \quad \text{for} \quad t < 0; \qquad \theta_i = Ct \quad \text{for} \quad t \geq 0. \tag{42}$$

Then $\epsilon(t)$ is a solution of the equation (40) made homogeneous. If the servomechanism is to start from rest at $t = 0$, the initial values are

$$\epsilon(0) = q_0 = q_0 = 0; \qquad \dot{\epsilon}(0) = q_1 = -C; \qquad \ddot{\epsilon}(0) = q_2 = 0. \tag{43}$$

Application of (35) leads to

$$2Y = C^2 \frac{a_1^2 + a_3 a_0^2}{a_3 D_2}. \tag{44}$$

It is obvious that Y becomes smaller with increasing a_2. Therefore a_2 should be made as large as possible. For practical reasons (saturation and overcontrol of amplifiers or the like) an upper bound for a_2 is given. With this in mind we minimize $2Y$ for a given a_2 by variation of a_3. Setting $b_i = a_i/a_0$ we find

$$\text{Min } 2Y = C^2 \frac{2b_1^2 + b_2 + 2b_1(b_1^2 + b_2)^{1/2}}{2b_1 b_2^2}, \tag{45}$$

$$b_3 = b_1^2\{(b_1^2 + b_2)^{1/2} - b_1\}. \tag{46}$$

This gives the best design with respect to the important case (42).

References

1. I. Schur, *Über algebraische Gleichungen, die nur Wurzeln mit negativen Realteilen besitzen*, Z. angew. Math. Mech. **1**, 307-311 (1921).
2. P. Hazebroek and B. I. van der Waerden, *Theoretical considerations on the optimum adjustment of regulators*, Trans. Amer. Soc. Mech. Engrs. **72**, 309-315 (1950).

VARIATIONAL PRINCIPLES OF EQUILIBRIUM OF AN ELASTO-PLASTIC BODY*

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In this paper the author attempts to establish the general forms of the variational principles of equilibrium of an elasto-plastic body, and to make clear the relations among the principles presented previously [1]-[6].

§1 Fundamental principle. Consider an elasto-plastic body[1] which is stressed by the surface traction F^i prescribed at each point on the portion S_F of the surface S of the body, the surface displacement v_i prescribed at each point on the remaining portion S_v of S, and the body force K^i prescribed throughout the interior V of the body. The stress and strain distributions in the body are assumed to be given by σ^{ij} and ϵ_{ij} respectively. Then we may select some incremental stress-strain law[2] to hold at each point in the body. We assume that this law is such that any small possible change of the stress-strain state satisfies the condition

$$\int_0^{\delta\epsilon} \delta\sigma^{ij}\, d\delta\epsilon_{ij} \geq 0\ ^3, \tag{1}$$

which is a generalized form of the so-called uniqueness condition [7].

Now, suppose that the increment ΔF^i of the surface traction F^i on S_F, the increment Δv_i of the surface displacement v_i on S_v and the increment ΔK^i of the body force K^i throughout V are added gradually to the body. Then the resulting distribution of stress, strain and incremental displacement becomes $\sigma^{ij} + \Delta\sigma^{ij}$, $\epsilon_{ij} + \Delta\epsilon_{ij}$ and Δu_i respectively. In the following lines we shall establish the variational principles which determine the resulting stress-strain state.

Since the resulting displacement and strain are usually small, we apply the infinitesimal deformation theory to our problem. An artificial displacement Δu_i^*, which is continuous and has piecewise continuous first and second derivatives, and which takes the prescribed value Δv_i on S_v, is called an admissible displacement. Corresponding to it we may determine the strain $\Delta\epsilon_{ij}^*$ by the equation

$$\Delta\epsilon_{ij}^* = \tfrac{1}{2}(\Delta u_{i,j}^* + \Delta u_{j,i}^*)\ ^4. \tag{2}$$

Then the fundamental principle may be stated as follows: The following expression is non-negative for any artificial admissible displacement process $\Delta u_i^*(t)$ $(t_0 \leqq t \leqq t_1)$

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[1] We apply the Green's theorem in this body, so that its surface must have a suitable regularity.

[2] We need not assume one to one correspondence of the increments of stress and strain.

[3] As usual the summation convention is used in this paper.

[4] Suffix ", j" is the sign of differentiation by the ordinate x^j.

such that $\Delta u_i^*(t_0) = \Delta u_i$, the actual displacement in the equilibrium state, and $\Delta u_i^*(t_1) = \Delta u_i^*$, an admissible displacement;

$$\iiint dV \int_{\Delta\epsilon}^{\Delta\epsilon^*} (\sigma^{ij} + \Delta\sigma^{*ij}(t))\, d\Delta\epsilon_{ij}^*(t) - \iiint dV\, \rho(K^i + \Delta K^i)(\Delta u_i^* - \Delta u_i) - \iint_{S_F} dS(F^i + \Delta F^i)(\Delta u_i^* - \Delta u_i)\,^5, \tag{3}$$

where ρ is the density of the material, the strain $\Delta\epsilon_{ij}^*(t)$ is derived by Eq. (2) from $\Delta u_i^*(t)$, and the stress $\Delta\sigma^{*ij}(t)$ is determined by $\Delta\epsilon_{ij}^*(t)$ and the stress-strain law.

The expression (3) is equivalent to

$$\iiint dV(\sigma^{ij} + \Delta\sigma^{ij})\, \delta\Delta\epsilon_{ij}^* - \iiint dV\, \rho(K^i + \Delta K^i)\, \delta\Delta u_i^* - \iint_{S_F} dS(F^i + \Delta F^i)\, \delta\Delta u_i^* + \iiint dV \int_0^{\delta\Delta\epsilon^*} \delta\Delta\sigma^{*ij}(t)\, d\delta\Delta\epsilon_{ij}^*(t), \tag{4}$$

where $\delta\Delta u_i^* = \Delta u_i^* - \Delta u_i$, etc. By the principle of virtual work the first three terms in this expression vanish. Furthermore, the last term is always non-negative by Eq. (1). Thus, our proposition is proved.

§2. **Minimum principles.** Unfortunately, the fundamental principle is not generally useful, because the value of the integral

$$\int_0^{\Delta\epsilon^*} (\sigma^{ij} + \Delta\sigma^{*ij}(t))\, d\epsilon_{ij}^*(t) \tag{5}$$

usually depends upon the stress-strain process. Hereafter we will assume that the value of this integral is determined only by the final values of $\Delta\sigma^{*ij}$ and $\Delta\epsilon_{ij}^*$ at each point in v, and that any stress-strain state is attainable from an arbitrary state. In such cases the following principle is clear.

Principle I: The expression

$$U + \Delta U^* \equiv \iiint dV \int_0^{\Delta\epsilon^*} (\sigma^{ij} + \Delta\sigma^{*ij}(t))\, d\Delta\epsilon_{ij}^*(t) - \iiint dV\, \rho(K^i + \Delta K^i)\, \Delta u_i^* - \iint_{S_F} dS\,(F^i + \Delta F^i)\, \Delta u_i^*$$

takes its minimum value when the admissible displacement coincides with the actual one.

Certainly, the first variation becomes,

$$\delta(U + \Delta U^*) = \iiint dV(\sigma^{ij} + \Delta\sigma^{*ij})\, \delta\Delta\epsilon_{ij}^* - \iiint dV\, \rho(K^i + \Delta K^i)\, \delta\Delta u_i^* - \iint_{S_F} dS(F^i + \Delta F^i)\, \delta\Delta u_i^*\,.$$

[5] Integration $\int_{\Delta\epsilon}^{\Delta\epsilon^*} \ldots d\Delta\epsilon_{ij}^*(t)$ is taken along the process of the strain $\Delta\epsilon_{ij}^*(t)$.

From this we obtain the well known equilibrium conditions of stress as follows:

$$(\sigma^{ij} + \Delta\sigma^{*ij})_{,j} + \rho(K^i + \Delta K^i) = 0, \text{ in } V, \tag{6}$$

except on the discontinuity surface S_d of stress in V,

$$[(\sigma_+^{ij} + \Delta\sigma_+^{*ij}) - (\sigma_-^{ij} + \Delta\sigma_-^{*ij})]m_j = 0, \text{ on } S_d\,, \tag{7}$$

where m_j is the unit normal vector on S_d, and σ_+^{ij} and σ_-^{ij} are the values of σ^{ij} on the both sides of S_d, etc., and

$$(\sigma^{ij} + \Delta\sigma^{*ij})n_j = F^i + \Delta F^i, \text{ on } S_F\,, \tag{8}$$

where n_j is the unit vector in the direction of the external normal of surface. The second variation becomes

$$\delta^2(U + \Delta U^*) = \iiint dV \int_0^{\delta\Delta\epsilon^*} \delta\Delta\sigma^{*ij}(t)\, d\delta\Delta\epsilon_{ij}^*(t).$$

Since by Eq. (1) this value is non-negative, $U + \Delta U^*$ takes its minimum value when the admissible displacement coincides with the actual one.

By (2) we may easily obtain the equality

$$\begin{aligned}\iiint dV\, \sigma^{ij}\, \Delta\epsilon_{ij}^* - \iiint dV\, \rho K^i\, \Delta u_i^* - \iint\limits_{S_F} dS\, F^i\, \Delta u_i^* \\ = \iint\limits_{S_v} dS\, \sigma^{ij} n_j\, \Delta v_i = \text{const.} \equiv U,\end{aligned} \quad \text{say.} \tag{9}$$

Then the following principle may be obtained by Principle I and Eq. (9).

Principle I′: The expression

$$\Delta U^* \equiv \iiint dV \int_0^{\Delta\epsilon^*} \Delta\sigma^{*ij}(t)\, d\Delta\epsilon_{ij}^*(t) - \iiint dV\, \rho\, \Delta K^i\, \Delta u_i^* - \iint\limits_{S_F} dS\, \Delta F^i\, \Delta u_i$$

takes its minimum value when the admissible state coincides with the actual one.

§3 Maximum principles. Since the incremental strain is given by Eq. (2), the involute transformation of Principles I and I′ may be easily obtained by the general method [8] [9].

In the first place, Principle I′ is equivalent to the following variational problem without any additional conditions,

$$\begin{aligned}\Delta U' \equiv \iiint dV \int_0^{\Delta\epsilon^{**}} \Delta\sigma^{**ij}(t)\, d\Delta\epsilon_{ij}^{**}(t) \\ - \iiint dV\, \rho\, \Delta K^i\, \Delta u_i^{**} - \iint\limits_{S_F} dS\, \Delta F^i\, \Delta u_i^{**} \\ - \iiint dV\, \lambda^{ij}\left[\Delta\epsilon_{ij}^{**} - \frac{1}{2}(\Delta u_{i,j}^{**} + \Delta u_{j,i}^{**})\right] \\ - \iint\limits_{S_v} dS\, \mu^i(\Delta u_i^{**} - \Delta v_i) = \text{Min.},\end{aligned} \tag{10}$$

where Δu_i^{**} is an artificial displacement which is continuous and piecewise differentiable, $\Delta \epsilon_{ij}^{**}$ is an artificial strain which is independent of Δu_i^{**} and piecewise continuous, $\Delta \sigma^{**ij}$ is the stress corresponding to $\Delta \epsilon_{ij}^{**}$, λ^{ij} and μ^i are Lagrangian multipliers, and especially λ^{ij} is piecewise continuous and differentiable. Then the natural conditions of Eq. (10) become

$$\left.\begin{aligned} \Delta\epsilon_{ij}^{**} &= \tfrac{1}{2}(\Delta u_{i,j}^{**} + \Delta u_{j,i}^{**}), \text{ in } V, \\ \Delta u_i^{**} &= \Delta v_i \text{ , on } S_v \text{ ,} \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} &\Delta\sigma^{**ij} = \lambda^{ij}, \text{ in } V, \lambda^{ij} n_j = \mu^i, \text{ on } S_v \text{ ,} \\ &\Delta F^i = \lambda^{ij} n_j \text{ , on } S_F \text{ ,} \\ &\lambda^{ij}_{,j} + \rho\, \Delta K^i = 0, \text{ in } V \text{ except on the discontinuity surface } S_d \text{ of } \lambda^{ij}, \text{ and} \\ &(\lambda^{ij}_+ - \lambda^{ij}_-) m_j = 0, \text{ on } S_d \text{ .} \end{aligned}\right\} \tag{12}$$

Since, if we add Eq. (11) to Eq. (10) as the additional conditions, Eq. (10) returns to Principle I′, we substitute λ^{ij} and μ^i into Eq. (10) from condition (12), i.e.,

$$\begin{aligned} \Delta U' \Big|_{(12)} &= -\iiint dV\left(\Delta\sigma^{**ij}\, \Delta\epsilon_{ij}^{**} - \int_0^{\Delta\epsilon^{**}} \Delta\sigma^{**ij}(t)\, d\Delta\epsilon_{ij}^{**}(t)\right) \\ &\qquad + \iint_{S_v} dS\, \Delta\sigma^{**ij}\, n_j v_i \\ &= -\iiint dV \int_0^{\Delta\sigma^{**}} \Delta\epsilon_{ij}^{**}(t)\, d\Delta\sigma^{**ij}(t) \\ &\qquad + \iint_{S_v} dS\, \Delta\sigma^{**ij}\, n_j\, \Delta v_i \equiv \Delta U^{**}, \qquad \text{say.} \end{aligned} \tag{13}$$

Now, the value of the integral

$$\int_0^{\Delta\sigma^{**}} \Delta\epsilon_{ij}^{**}(t)\, d\Delta\sigma^{**ij}(t) = \Delta\sigma^{**ij}\, \Delta\epsilon_{ij}^{**} - \int_0^{\Delta\epsilon^{**}} \Delta\sigma^{**ij}(t)\, d\Delta\epsilon_{ij}^{**}(t) \tag{14}$$

is also determined by the final values of $\Delta\sigma^{**ij}$ and $\Delta\epsilon_{ij}^{**}$ only. Accordingly, we may regard $\Delta\sigma^{**ij}$ as the independent variable in the expression (13). When $\Delta\sigma^{**ij}$ is piecewise differentiable and satisfies the equilibrium conditions (6), (7), and (8),[6] we call such a stress $\Delta\sigma^{**ij}$ an admissible stress. Then, by the general theory of the involute transformation the following principle is valid.

Principle II′: The expression

$$\Delta U^{**} \equiv -\iiint dV \int_0^{\Delta\sigma^{**}} \Delta\epsilon_{ij}^{**}(t)\, d\Delta\sigma^{**ij}(t) + \iint_{S_v} dS\, \Delta\sigma^{**ij}\, n_j\, \Delta v_i$$

takes its maximum value when the admissible stress coincides with the actual stress.

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[6]Because the conditions (12) correspond to the equilibrium condition.

Certainly, we may confirm this principle as follows: Since, putting $\Delta\sigma^{**ij} - \Delta\sigma^{ij} = \delta\Delta\sigma^{**ij}$, it is clear that, $\delta\Delta\sigma^{**ij}n_j = \delta F^i = 0$, on S_F,

$$\delta\Delta\sigma^{**ij}_{;j} = -\delta(\rho K^i) = 0, \text{ in } V,$$

except on the discontinuity surface S_d of $\delta\Delta\sigma^{**ij}$, and

$$(\delta\Delta\sigma^{**ij}_{+} - \delta\Delta\sigma^{**ij}_{-})m_j = 0, \text{ on } S_d,$$

we obtain the equality

$$0 = \iiint dV\, \Delta u_i^{**}\, \delta\Delta\sigma^{**ij}_{;j} = -\iiint dV\, \Delta u^{**}_{i;j}\, \delta\Delta\sigma^{**ij} + \iint_{S_v} dS\, \Delta u_i^{**}\, n_j\, \delta\Delta\sigma^{**ij} \quad (15)$$

for any function Δu_i^{**} which is continuous and has piecewise continuous first derivatives. Then the first variation of ΔU^{**} becomes,

$$\begin{aligned}\delta\Delta U^{**} &= -\iiint dV\, \Delta\epsilon^{**}_{ij}\, \delta\Delta\sigma^{**ij} + \iint_{S_v} dS\, \delta\Delta\sigma^{**ij}\, n_j\, \Delta v_i \\ &= -\iiint dV(\Delta\epsilon^{**}_{ij} - \Delta u^{**}_{i;j})\, \delta\Delta\sigma^{**ij} + \iint_{S_v} dS\, \delta\Delta\sigma^{**ij}\, n_j(\Delta v_i - \Delta u_i^{**}).\end{aligned}$$

That is, as the natural condition of this principle, we obtain the fact that $\Delta\epsilon^{**}_{ij}$ may be derived by Eq. (2) from suitable possible displacement Δu_i^{**}. Furthermore the second variation of ΔU^{**} becomes,

$$\begin{aligned}\delta^2\Delta U^{**} &= -\iiint dV\Big[\int_0^{\Delta\sigma^{**}} \Delta\epsilon^{**}_{ij}(t)\, d\Delta\sigma^{**ij}(t) \\ &\qquad - \int_0^{\Delta\sigma} \Delta\epsilon_{ij}(t)\, d\Delta\sigma^{ij}(t) - (\Delta\sigma^{**ij} - \Delta\sigma^{ij})\, \Delta\epsilon_{ij}\Big] \\ &= -\iiint dV \int_0^{\delta\Delta\sigma^{**}} \delta\Delta\epsilon^{**}_{ij}(t)\, d\delta\Delta\sigma^{**ij}(t).\end{aligned}$$

Since, we may attain to the stress-strain state $(\Delta\sigma^{ij}, \Delta\epsilon_{ij})$ from the state $(\Delta\sigma^{**ij}, \Delta\epsilon^{**}_{ij})$ by suitable process, we obtain the equalities

$$\begin{aligned}\int_0^{\delta\Delta\sigma^{**}} \delta\Delta\epsilon^{**}_{ij}(t)\, d\delta\Delta\sigma^{**ij}(t) &= \delta\Delta\sigma^{**ij}\, \delta\Delta\epsilon^{**}_{ij} - \int_0^{\delta\Delta\epsilon^{**}} \delta\Delta\sigma^{**ij}(t)\, d\delta\Delta\epsilon^{**}_{ij}(t) \\ &= \int_0^{-\delta\Delta\epsilon^{**}} \delta'\Delta\sigma^{**ij}(t')\, d\delta'\Delta\epsilon^{**}_{ij}(t'),\end{aligned}$$

where

$$\delta'\Delta\epsilon^{**}_{ij}(t') = \Delta\epsilon^{**}_{ij}(-t') - \Delta\epsilon^{**}_{ij}.$$

Accordingly, by condition (1) the value of $\delta^2\Delta U^{**}$ is always non-positive. This shows that ΔU^{**} generally takes its maximum value when the admissible stress coincides with the actual stress.

By Eq. (9), we may obtain the following principle.

Principle II: The expression

$$U + \Delta U^{**} = -\iiint dV \int_0^{\Delta\sigma^{**}} \Delta\epsilon_{ij}^{**}(t)\, d\Delta\sigma^{**ij}(t) + \iint_{S_v} dS(\sigma^{ij} + \Delta\sigma^{**ij})\, n_j\, \Delta v_i$$

takes its maximum value when the admissible stress coincides with the actual.

From the general character of the involute transformation the following relations are obvious,

$$\left.\begin{aligned} \text{Min. } \Delta U^* &= \text{Max. } \Delta U^{**}, \qquad \text{and} \\ \text{Min. } (U + \Delta U^*) &= \text{Max. } (U + \Delta U^{**}) = U + \text{Min. } \Delta U^*. \end{aligned}\right\} \tag{16}$$

Certainly,

$$\begin{aligned} \text{Min. } \Delta U^* &= \Delta U^* \Big|_{\Delta u_i^* = \Delta u_i} \\ &= \iiint dV \int_0^{\Delta\epsilon} \Delta\sigma^{ij}(t)\, d\Delta\epsilon_{ij}(t) - \iiint dV\, \rho\, \Delta K^i\, \Delta u_i - \iint_{S_F} dS\, \Delta F^i\, \Delta u_i \\ &= \iiint dV\, \Delta\sigma^{ij}\, \Delta\epsilon_{ij} - \iiint dV\, \rho\, \Delta K^i\, \Delta u_i - \iint_{S_F} dS\, \Delta F^i\, \Delta u_i \\ &\qquad - \iiint dV\left(\Delta\sigma^{ij}\, \Delta\epsilon_{ij} - \int_0^{\Delta\epsilon} \Delta\sigma^{ij}(t)\, d\Delta\epsilon_{ij}(t)\right) \\ &= \iint_{S_v} dS\, \Delta\sigma^{ij}\, n_j\, \Delta v_i - \iiint dV \int_0^{\Delta\sigma} \Delta\epsilon_{ij}(t)\, d\Delta\sigma^{ij}(t) \\ &= \Delta U^{**} \Big|_{\Delta\sigma^{**ij} = \Delta\sigma^{ij}} = \text{Max. } \Delta U^{**}. \end{aligned}$$

§4 When are our principles valid? As already described our variational principles are valid when and only when the values of the integrals (5) and (14) are independent of the stress-strain process. Accordingly, when and only when the value of the integral

$$\int_0^{\epsilon} \sigma^{ij}\, d\epsilon_{ij} \tag{17}$$

is determined by the final values of stress and strain only, our principles are valid.

There are three reasons why the value of the integral (17) depends upon the stress-strain process: The first is the dependence of the stress-strain law upon the plastic history. Accordingly, we must assume that the stress-strain law is independent of the instantaneous plastic history. The second is the irreversibility of the stress-strain relation in case of the plastic deformation. That is, the stress-strain relations for loading and unloading states are individual ones. Then, our principles are valid for the cases where we may expect that the actual incremental process does not contain any unloading process at any yield surfaces except the initial one. Because in such cases we need not adopt the artificial admissible processes which contain such unloading processes. The third is the lack of the complete integrability of the integral (17) in stress space. We may easily obtain the general form of the stress-strain law which makes the integral

(17) completely integrable if the unloading processes do not occur, and it becomes as follows:

$$d\epsilon_{ij} = E_{ijkl}\, d\sigma^{kl} + F(f)\, \frac{\partial f}{\partial \sigma^{ij}}\, (df + |\, df\, |)\left(2\sigma^{pq}\, \frac{\partial f}{\partial \sigma^{pq}}\right)^{-1}, \tag{18}$$

where $E_{ijkl} = E_{klij}$ is the elastic modulus, $f(\sigma)$ is the so-called loading function which is a function of σ^{ij} only and the domain $f <$ const. $= c$ is the instantaneous elastic domain, and $F(f)$ is a positive function of $f(\sigma)$ only [10]. In this paper we assume such a stress-strain law.[7]

As above described, even if the stress strain law is given by Eq. (18), we can not assert the validity of our principles. But, if we can foresee that the stress-strain process, by which the resulting stress-strain state is attained, does not contain any unloading processes at any yield surfaces except the initial one $f(\sigma) = c_0$, where c_0 is determined by the plastic history just before the incremental deformation, we may form our principles for the admissible states which are attained by such processes. If we assume the following reversible stress-strain law which satisfies the condition (1), we may form such principles,

$$\begin{aligned} d\Delta\epsilon_{ij} &= E_{ijkl}\, d\Delta\sigma^{kl}, && \text{wherever } f < c_0 \text{, or } f = c_0 \text{ but } df \leq 0,\\ &= E_{ijkl}\, d\Delta\sigma^{kl} + F(f)\, \frac{\partial f}{\partial \sigma^{ij}}\, \frac{\partial f}{\partial \sigma^{kl}}\, d\Delta\sigma^{kl}\left[(\sigma^{pq} + \Delta\sigma^{pq})\, \frac{\partial f}{\partial \sigma^{pq}}\right]^{-1}, \\ & && \text{wherever } f > c_0 \text{, or } f = c_0 \text{ but } df > 0, \end{aligned}$$

where $\quad f = f(\sigma^{ij} + \Delta\sigma^{ij})$.

The solution state of such principles satisfies the equilibrium and compatibility conditions, and by our assumption it is attainable by a suitable process whose stress-strain relation follows the Eq. (18), and which does not contain any unloading processes except those on the yield surface $f = c_0$. That is, in this case the actual resulting state is governed by these principles.

§5 Principles for the stress and strain rates. Usually the mechanical quantities, such as strain, vary with finite time rates during a gradual stressing. In such cases they may be regarded to relate linearly with time during small time interval τ, i.e.,

$$\Delta\epsilon_{ij}^*(t) = \dot{\epsilon}^*(t), \qquad \text{etc.,} \qquad (0 \leq t < \tau)$$

except in the very small region whose volume vanishes with τ. In the limiting case where τ tends to zero, we may regard that the process, by which the resulting state is attained, does not contain any unloading process except those on the initial yield surface. Then, as already described, our principles may be formed as principles for the rates. That is, they are easily obtained from Principles I′ and II′ as follows:

The expression

$$\text{I}'_a: \qquad \frac{1}{2}\, \ddot{U}^* = \iiint dV\, \frac{1}{2}\, \dot{\sigma}^{*ij}\dot{\epsilon}_{ij}^* - \iiint dV\, \rho \dot{K}^i \dot{u}_i^* - \iint_{S_F} dS\, \dot{F}^i \dot{u}_i^*$$

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[7]This is a generalized form of the Hodge-Prager stress-strain law [1].

takes its minimum value when the admissible displacement rate $\dot{u}_i^*$ coincides with the actual one.

The expression

$$\text{II}_a': \qquad \frac{1}{2}\ddot{U}^{**} = -\iiint dV \frac{1}{2}\dot{\sigma}^{**ij}\dot{\epsilon}_{ij}^{**} + \iint_{S_v} dS\, \dot{\sigma}^{**ij} n_j \dot{v}_i$$

takes its maximum value when the admissible stress rate $\dot{\sigma}_{ij}^{**}$ coincides with the actual one.

The principle of Hodge and Prager is equivalent to our Principle II_a' [1].

We may write down our principles for specific incremental stress-strain laws which belong to our general form (18). For example in the following lines we shall establish the principles for the material whose stress-strain relation follows the Prandtl-Reuss law,[8]

$$\dot{\epsilon}_{ij} = (2G_0)^{-1}\dot{s}^{ij} + \mu s^{ij},$$

where

$$\dot{s}^{ij} = \dot{\sigma}^{ij} - \frac{1}{3}\delta^{ij}\dot{\sigma}^{kk}, \qquad J_2 = \frac{1}{2}s^{ij}s^{ij},$$

and
$$\mu = \begin{cases} 0 & \text{wherever } J_2 < k^2, \text{ or } J_2 = k^2 \text{ but } \dot{J}_2 < 0, \\ (2k^2)^{-1}s^{ij}\dot{\epsilon}_{ij} > 0, & \text{wherever } J_2 = k^2 \text{ and } \dot{J}_2 = 0. \end{cases}$$

In this case the following relations are easily obtained,

$$\dot{\epsilon}_{ii} = 0, \qquad \sigma^{ij}\dot{\epsilon}_{ij} = s^{ij}\dot{\epsilon}_{ij} = (2G)^{-1}\dot{s}^{ij}s^{ij} + \mu s^{ij}s^{ij},$$
$$\dot{\sigma}^{ij}\dot{\epsilon}_{ij} = 2G_0(\dot{\epsilon}_{ij}\dot{\epsilon}_{ij} - \mu s^{ij}\dot{\epsilon}_{ij}) = (2G_0)^{-1}\dot{s}^{ij}\dot{s}^{ij} \geq 0.$$ [9]

Then our principles become as follows:

The expression

$$\text{I}_b': \quad \frac{1}{2}\ddot{U}^* = \iiint dV\, G_0(\dot{\epsilon}_{ij}^*\dot{\epsilon}_{ij}^* - \mu^* s^{ij}\dot{\epsilon}_{ij}^*) - \iiint dV\, \rho\dot{K}^i\dot{u}_i^* - \iint_{S_F} dS\, \dot{F}^i\dot{u}_i^*,$$

where

$$\mu^* = \begin{cases} 0, & \text{wherever } J_2 < k^2, \text{ or } J_2 = k^2 \text{ but } s^{ij}\dot{\epsilon}_{ij}^* \leq 0, \\ (2k^2)^{-1}S^{ij}\dot{\epsilon}_{ij}^*, & \text{wherever } J_2 = k^2 \text{ and } s^{ij}\dot{\epsilon}^* > 0 \end{cases}$$

and the expression

$$\text{II}_b': \qquad \frac{1}{2}\ddot{U}^{**} = -\iiint dV(4G_0)^{-1}\dot{S}^{**ij}\dot{s}^{**ij} + \iint_{S_v} dS\, \dot{\sigma}^{**ij}n_j\dot{v}_i$$

take on minimum and maximum values respectively when the admissible rates coincide with the actual ones.

[8]The Prandtl-Reuss law and the Levy-Mises law belong to the Hodge-Prager's general type as the extreme cases [1].

[9]This condition corresponds to condition (1).

The principles of Greenberg are equivalent to these principles [2] [3].

In the case where the elastic strain is very small compared with the plastic strain, we may neglect the elastic strain in the stress-strain law. In such cases our problem becomes somewhat complicated, because the stress state is not uniquely determined by a strain state. But the essential properties of our problem are unaltered by such circumstances. For example, we derive the principles for the material whose stress-strain relation follows the Levy-Mises law[8] i.e.,

$$d\epsilon_{ij} = \lambda s^{ij},$$

where

$$\lambda = \begin{cases} 0, & \text{wherever } J_2 < k^2, \text{ or } J_2 = k^2 \text{ but } dJ_2 < 0 \\ (2k^2)^{-1} s^{ij}\, d\epsilon ij\,, & \text{wherever } J_2 = k^2 \text{ and } dJ_2 = 0. \end{cases}$$

In this case the increments of displacement and strain increase linearly with time, but the stress change is independent of time, i.e.,

$$\Delta\epsilon^*_{ij} = \dot{\epsilon}^*_{ij}(t), \qquad \Delta\sigma^{*ij} = \text{const., etc.,} \qquad (0 < t \leq \tau).$$

Then the following relations are obtained,

$$d\epsilon_{ij} = s^{ij}[d\epsilon_{pq}\, d\epsilon_{pq}(2k^2)^{-1}]^{1/2},$$

$$\int_0^{\Delta\epsilon^*} (\sigma^{ij} + \Delta\sigma^{*ij}(t))\, d\Delta\epsilon^*_{ij}(t) = (\sigma^{ij} + \Delta\sigma^{*ij})\, \Delta\epsilon^*_{ij} = (2k^2\, \Delta\epsilon^*_{pq}\, \Delta\epsilon^*_{pq})^{1/2}$$

$$\int_0^{\Delta\sigma^{**}} \Delta\epsilon^{**}_{ij}(t)\, d\Delta\sigma^{**ij}(t) = 0,$$

$$\int_0^{\delta\epsilon} \delta\sigma^{ij}(t)\, d\delta\epsilon_{ij}(t) = \delta\sigma^{ij}\, \delta\epsilon_{ij} = \delta s^{ij} s^{ij}[\delta\epsilon_{pq}\, \delta\epsilon_{pq}(2k^2)^{-1}]^{1/2} \geq 0. \text{ [9]}$$

Then our principles are easily obtained by Principles I and II as follows:

The expression

$$I_c: \qquad \dot{U}^* = \iiint dV (2k^2 \dot{\epsilon}_{pq} \dot{\epsilon}_{pq})^{1/2} - \iiint dV\, \rho K^i \dot{u}^*_i - \iint_{S_F} dS\, F^i \dot{u}^*_i$$

and the expression

$$\mathrm{II}_c: \qquad \dot{U}^{**} = \iint_{S_u} dS (\sigma^{ij} + \Delta\sigma^{**ij}) n_j \dot{v}_i$$

take on minimum and maximum values respectively when the admissible displacement rate $\dot{u}^*_i$ and the admissible stress $\Delta\sigma^{**ij}$ coincide with the actual ones.

The principle of Markov and Hill and the principle of Sadowsky are equivalent to I_c and II_c respectively [4].

§6 **Conclusion.** The fundamental principle and Principle I are valid even in the case of finite deformations, but the other principles have an essential restriction in condition (2).[10]

[10] The Phillips' second principle seems to be dubious in some respects [6]. Involute transformation for finite deformation becomes very complicated.

By the thermodynamical method we may prove the fundamental principle for stable equilibrium. Then the condition (1) seems to be essential in case of stable equilibrium.[11] Moreover, it is notable that usually the condition (1) is equivalent to the uniqueness condition.

As described above our minimum and maximum principles are valid only in the restricted case. Accordingly, for the general cases we must apply the principle of virtual work directly, instead of the variational principle.

Acknowledgement

The author wishes to express his thanks to Professor W. Prager, Professor M. Yoshiki and Assistant Professor T. Kanazawa for constructive suggestions and criticism.

Bibliography

[1] P. Hodge and W. Prager, *A variational principle for plastic material with strain-hardening*, J. Math Phys. 27, 4 (1949).
[2] H. J. Greenberg, *Complementary minimum principles for an elastic-plastic material*, Q. Appl. Math., 7, 1 (1949).
[3] W. Prager, *Fundamental theorem of a new mathematical theory of plasticity*, Duke Math. J., No. 1 (1942).
[4] R. Hill, *A comparative study of some variational principles in the theory of plasticity*, J. Appl. Mech., March (1950).
[5] A. H. Philippidis, *The general proof of the principle maximum plastic resistance*, J. Appl. Mech., 15, 3 (1948).
[6] A. Phillips, *Variational principle in the theory of finite plastic deformation*, Q. Appl. Math., 7, 1 (1949).
[7] W. Prager, *Recent development in the mathematical theory of plasticity*, J. Appl. Phys., 20, 3 (1949).
[8] R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, 1 ter Bd., 2 te Auf., Kap. 4, 9, Julius Springer, Berlin (1931).
[9] M. Kotani, *Mechanics of the continuous body*, Lecture on Mathematics. Iwanami, Tokyo (in Japanese) (1939).
[10] Y. Yamamoto, *A general theory on the plastic behavior of metals*, to be published in Proc. First N.C.T.A.M. in Japan.

[11]If the condition (1) fails, new instability phenomena may occur. It seems to the author that it corresponds to fracture or yield.

DIFFICULTIES WITH PRESENT SOLUTIONS OF THE HALLÉN INTEGRAL EQUATION*

BY

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I. Introduction. In a recent paper,[1] rather serious discrepancies were shown to exist between values of broadside absorption gain and back-scattering cross section as found from experiment, and those predicted from Hallén's first-order solution[2] as modified by King and Middleton.[3] In this paper these discrepancies and certain additional shortcomings with the present solutions to Hallén's integral equation will be discussed.

II. The first-order current distributions. It can be shown that the current distribution on a receiving dipole antenna is given by[4]

$$I_r(z) = I_E(z) - I_v(z), \tag{1}$$

where $I_E(z)$ is the current distribution due to the external field, E_i, on the antenna with zero load (shorted), and $I_v(z)$ is the current distribution along the antenna when driven by a voltage, V_L, equal to the voltage drop across the receiving antenna load. The distribution of Eq. (1) need be considered only in the cases where scattering behavior is desired. All the other properties of a receiving antenna usually of interest, such as absorption gain, impedance, and effective length, are determined by the driven current distribution alone.

The two different current distributions involved are given as follows:

a. The transmitting dipole. The first-order solution of Hallén's integral equation for the current distribution on a center-fed dipole of length $2h$ and radius a is given by

$$I_v(z) = j2\pi V_0 f_v(z)(\zeta\psi_t H_2)^{-1}, \tag{2}$$

where V_0 is the voltage at the terminals, $\zeta = 120\pi$ ohms, and

$$f_v(z) = f_v'(z) + jf_v''(z) = b_1 \cos\beta z - b_2 \sin\beta\,|\,z\,| - C(z)\sin\beta h + S(z)\cos\beta h, \tag{3a}$$

$$b_1 = [2\psi_t + E(h)]\sin\beta h - S(h), \tag{3b}$$

$$b_2 = [2\psi_t + E(h)]\cos\beta h - C(h), \tag{3c}$$

$$H_2 = H_2' + jH_2'' = [\psi_t + E(h)]\cos\beta h - C(h). \tag{4}$$

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*Received September 25, 1951.

[1] S. H. Dike and D. D. King, *The cylindrical dipole receiving antenna*, Tech. Report No. 12, Radiation Laboratory, Johns Hopkins University, 1951. (Submitted to *Proc. of I.R.E.* for publication).

[2] E. Hallén, *Theoretical investigations into the transmitting and receiving properties of antennas*, Nova Acta, Royal Soc. Sciences (Uppsala) 11, 1-44 (1938).

[3] R. W. P. King and D. Middleton, *The cylindrical antenna: current and impedance*, Q. Appl. Math. 3, 302-335 (1946).

[4] R. W. P. King, H. Mimno, and A. Wing, *Transmission lines, antennas and wave guides*, McGraw-Hill Book Co., New York, p. 163; 1945.

The functions $C(z)$, $S(z)$ and $E(z)$ are defined in Ref. 3. The function ψ_t is the expansion parameter. It is determined in the King-Middleton method by considering a function defined by

$$\Psi_t(z) = \int_{-h}^{h} g_t(z, s) r^{-1} e^{-j\beta r}\, ds, \tag{5}$$

where

$$r = [(z - s)^2 + a^2]^{1/2}, \tag{6}$$

and

$$g_t(z, s) = f(s)/f(z), \tag{7}$$

such that

$$I_v(z) = I_{0v} f(z);\ I_v(s) = I_{0v} f(s), \tag{8}$$

where I_{0v} is the terminal current in the driven dipole. In the limit of vanishing dipole radius the driven dipole current distribution can be shown to be[5]

$$I_v(z) = I_{0v}\,(\sin \beta h)^{-1} \sin \beta(h - |z|). \tag{9}$$

King and Middleton choose the function $f(z) = \sin \beta(h - |z|)$ giving

$$g_t(z, s) = \sin \beta(h - |s|)/\sin \beta(h - |z|), \tag{10}$$

and

$$\Psi_t(z) = \frac{C(z) \sin \beta h - S(z) \cos \beta h}{\sin \beta(h - |z|)}. \tag{11}$$

King and Middleton then argue that $\Psi_t(z)$ is predominately real, and that a suitable expansion parameter may be found by setting

$$\Psi_t(z) = |\Psi_t(z_0)| = \psi_t\,, \tag{12}$$

where z_0 is chosen so that $\Psi_t(z_0)$ is a good approximation to $\Psi_t(z)$ over most of the antenna. Accordingly, they choose

$$\psi_t = \begin{cases} |C(0) \sin \beta h - S(0) \cos \beta h|\,(\sin \beta h)^{-1}, & \beta h \le \dfrac{\pi}{2}, \\[2ex] \left| C\left(h - \dfrac{\lambda}{4}\right) \sin \beta h - S\left(h - \dfrac{\lambda}{4}\right) \cos \beta h \right|, & \beta h \ge \dfrac{\pi}{2}. \end{cases} \tag{13}$$

b. The unloaded receiving dipole. The first-order current distribution on a shorted dipole antenna placed parallel to the electric vector of a plane-wave, far-zone field is

$$I_E(z) = j4\pi E_i f_E(z) (\beta \zeta \psi_r H_1)^{-1}, \tag{14}$$

where

$$f_E(z) = 2\psi_r \cos \beta z + E(z) \cos \beta h - C(z) + C(h) - \cos \beta h[2\psi_r + E(h)], \tag{15}$$

$$H_1 = H_1' + jH_1'' = \psi_r \cos \beta h + E(h) \cos \beta h - C(h). \tag{16}$$

[5]S. Schelkunoff, *Electromagnetic waves*, D. Van Nostrand Co., New York, 1943, p. 142.

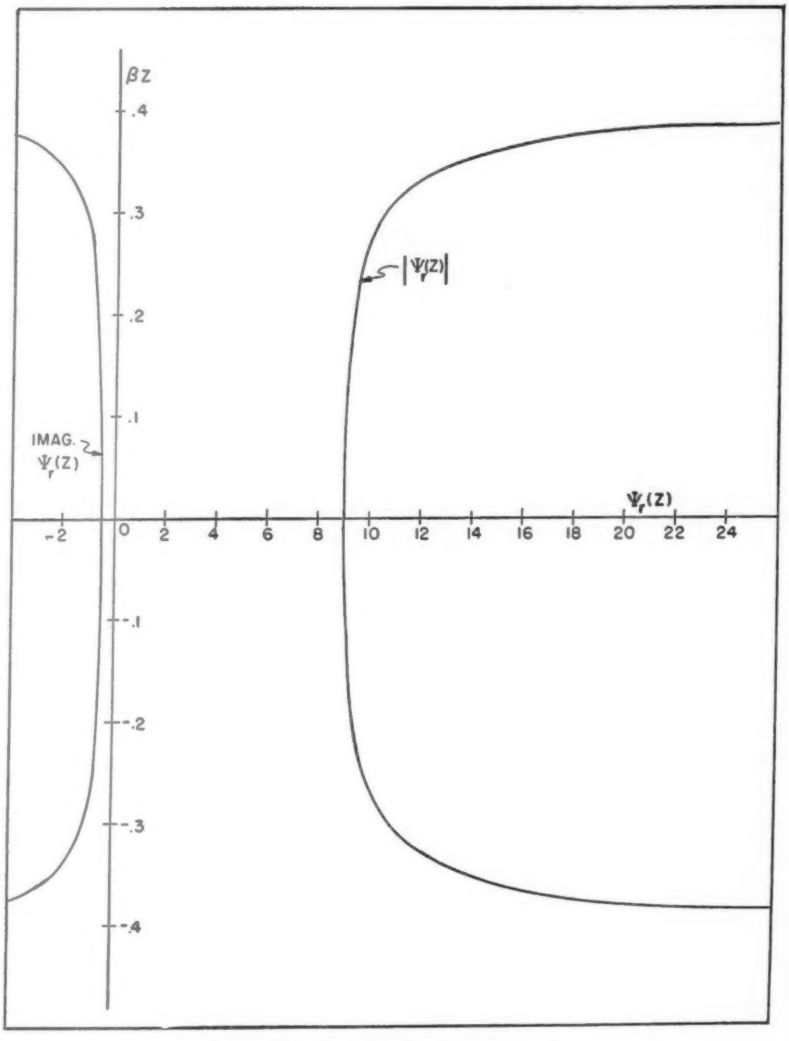

FIG. 1. The Function $\Psi_r(z)$ for $\beta h = 0.4$.
$\Omega = 10$

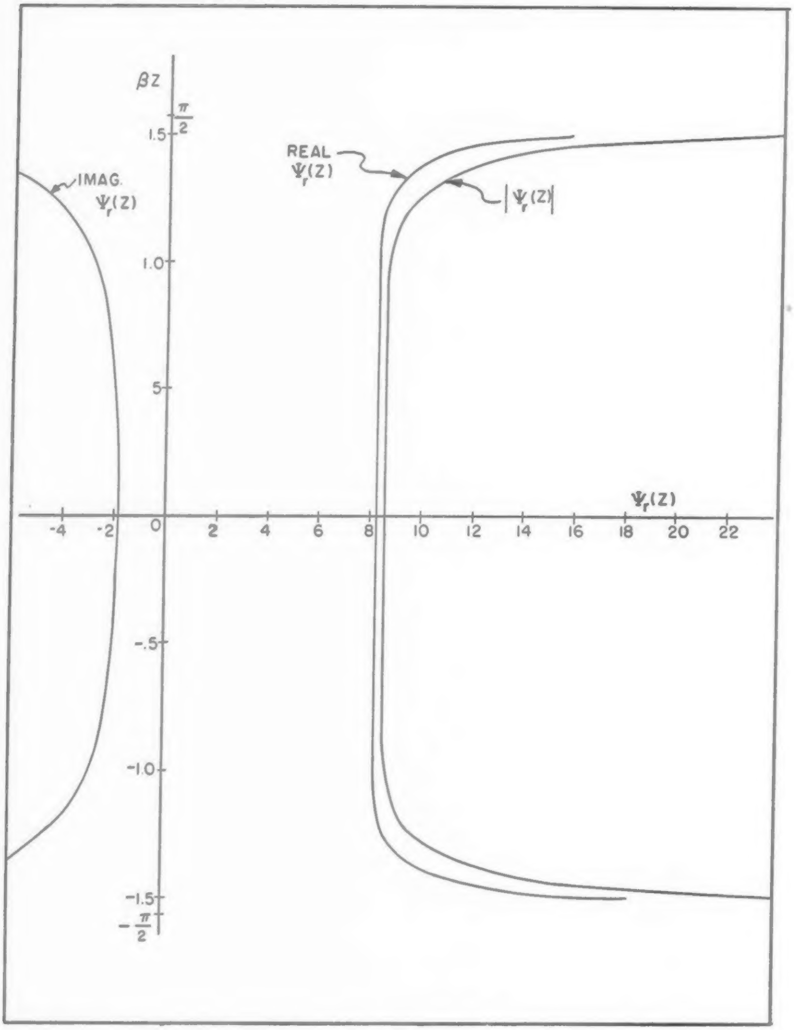

FIG. 2. The Function $\Psi_r(z)$ for $\beta h = \pi/2$
$\Omega = 10$

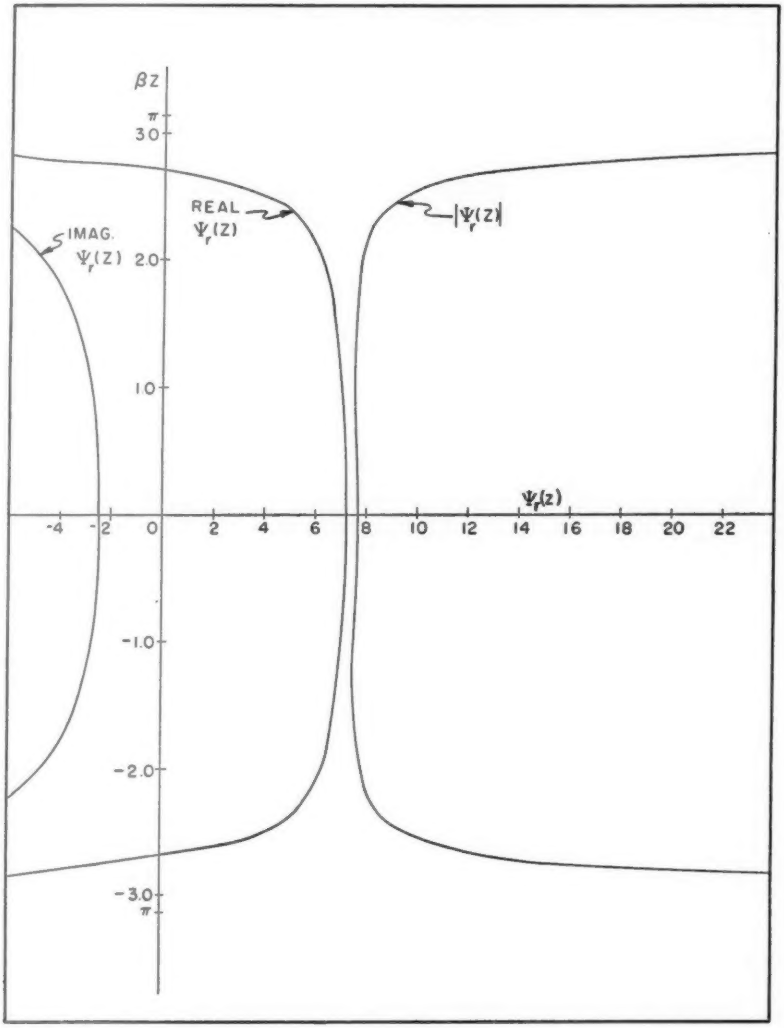

FIG. 3. The Function $\Psi_r(z)$ for $\beta h = \pi$
$\Omega = 10$

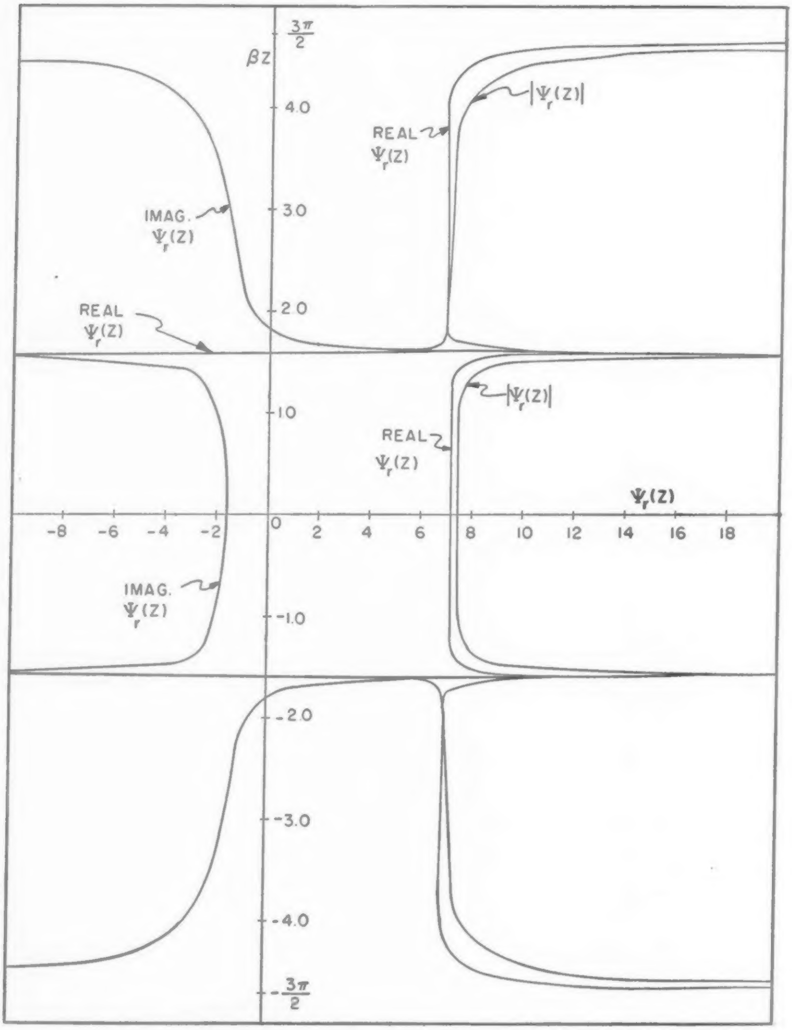

FIG. 4. The Function $\Psi_r(z)$ for $\beta h = \frac{3\pi}{2}$
$\Omega = 10$

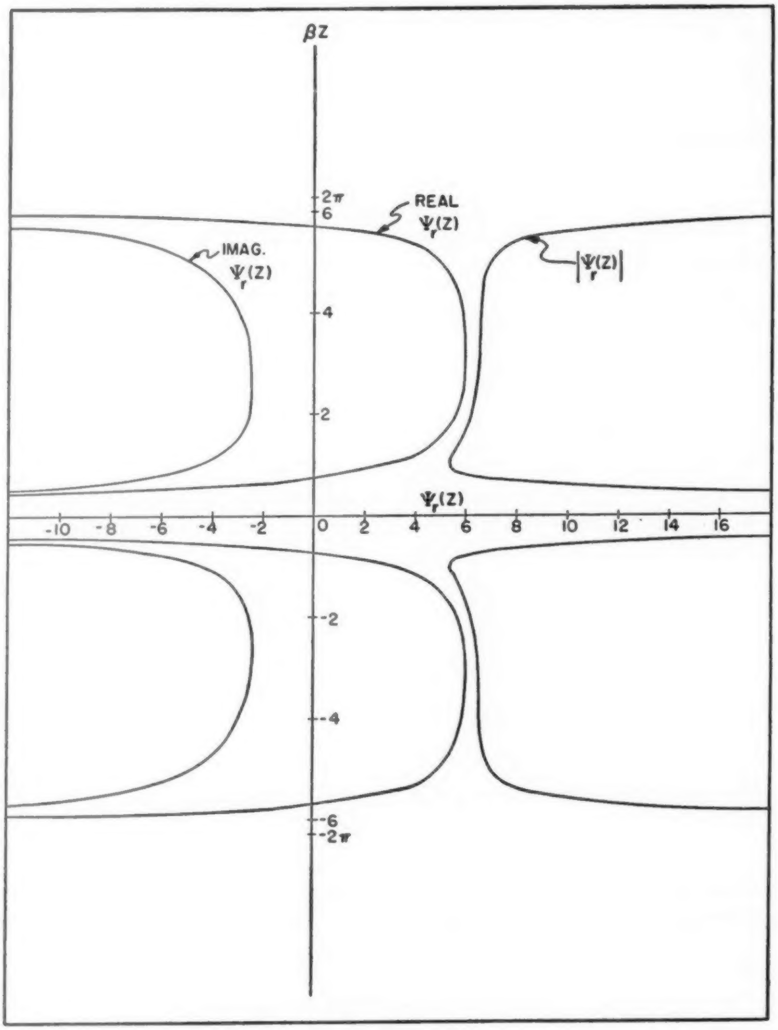

FIG. 5. The Function $\Psi_r(z)$ for $\beta h = 2\pi$
$\Omega = 10$

For the indefinitely thin shorted dipole, the current distribution is given by[6]

$$I_E(z) = I_{0E}(1 - \cos \beta h)^{-1}(\cos \beta z - \cos \beta h). \tag{17}$$

By the arguments of King and Middleton one is led to choose $f(z) = \cos \beta z - \cos \beta h$ so that

$$g_r(z, s) = \frac{\cos \beta s - \cos \beta h}{\cos \beta z - \cos \beta h}, \tag{18}$$

and $\Psi_r(z)$ becomes

$$\Psi_r(z) = \frac{C(z) - E(z) \cos \beta h}{\cos \beta z - \cos \beta h}. \tag{19}$$

This function is plotted in Figs. 1 to 5 for various values of βh and for $\Omega = 2 \log 2h/a = 10$. By consideration of these figues and using the arguments of King and Middleton, one finds that a suitable choice of ψ_r is

$$\psi_r = \begin{cases} \mid C(0) - E(0) \cos \beta h \mid (1 - \cos \beta h)^{-1}, & \beta h \leq \pi, \\ \dfrac{1}{2} \mid C(0) + E(0) \mid, & \beta h \geq \pi. \end{cases} \tag{20}$$

In the limit of very small βh, $\psi_t = \Omega - 2$ and $\psi_r = \Omega - 1$.

For the broadside case both the currents $I_v(z)$ and $I_E(z)$ are even functions. Equation (1) may now be expressed in terms of the foregoing as

$$I_r(z) = \frac{j4\pi E_i}{\beta \zeta \psi_r H_1} \left\{ f_E(z) - \frac{j2\pi Z_e f_E(0) f_v(z)}{\zeta \psi_t H_2} \right\}, \tag{21}$$

where

$$Z_e = Z_a Z_L/(Z_a + Z_L). \tag{22}$$

Z_a is the antenna impedance defined by

$$Z_a = V_0/I_{0v} . \tag{23}$$

Z_L is the receiving antenna load impedance. For matched load

$$Z_e = \mid Z_a \mid^2/2R_a , \qquad Z_L = Z_a^* . \tag{24}$$

III. Formulas for broadside gain, effective length and back-scattering cross section. Three equivalent expressions for absorption gain were derived in Ref. 1. They are

$$G_t = \frac{\pi \mid Z_a \mid^2 \mid \beta g_v(h) \mid^2}{\zeta \psi_t^2 R_a \mid H_2 \mid^2}, \tag{25}$$

$$G_t = \frac{\mid \beta g_v(h) \mid^2}{2\psi_t T}, \tag{26}$$

$$G_t = \frac{\pi \zeta \mid d \mid^2}{R_a \lambda^2}, \tag{27}$$

[6] King, Mimno, and Wing, *loc. cit.*

where

$$T = H_2''f_v'(0) - H_2'f_v''(0), \tag{28}$$

and where, for the broadside case,*

$$\begin{aligned}\beta g_v(h) = \beta \int_{-h}^{h} f_v(z)\, dz &= (4\psi_t - 2\Omega)(1 - \cos \beta h) \\ &+ 2 \operatorname{Ein} 2\beta h + \cos \beta h(\operatorname{Ein} 2\beta h - \operatorname{Ein} 4\beta h - 4 \operatorname{Ein} \beta h) - j \sin \beta h \operatorname{Ein} 4\beta h.\end{aligned} \tag{29}$$

R_a is the radiation resistance defined as the real part of Eq. (23). The symbol d denotes the effective length of the antenna. For the broadside case[7]

$$d = \frac{1}{I_{0v}} \int_{-h}^{h} I_v(z)\, dz, \tag{30}$$

or

$$| d |/\lambda = | \beta g_v(h) | (2\pi | f_v(0) |)^{-1}, \tag{31}$$

or

$$| d |/\lambda = | Z_a | | \beta g_v(h) | (\zeta\psi_t | H_2 |)^{-1}. \tag{32}$$

The relation for the back-scattering cross section, σ, derived in Ref. 1 is

$$\sigma/\lambda^2 = | \beta g_E(h) - BZ_e f_E(0) \beta g_v(h) |^2 (\pi\psi_r^2 | H_1 |^2)^{-1}, \tag{33}$$

where

$$B = j2\pi(\zeta\psi_t H_2)^{-1} = 2\pi(H_2'' + jH_2')(\zeta\psi_t | H_2 |^2)^{-1}, \tag{34}$$

and

$$\begin{aligned}\beta g_E(h) = \beta \int_{-h}^{h} f_E(z)\, dz &= \sin \beta h(4\psi_r - 2\Omega - 4 \log 2 \\ &- j \operatorname{Ein} 4\beta h + 2 \operatorname{Ein} 2\beta h + \operatorname{Ein} 4\beta h) \\ &+ \cos \beta h[2\beta h(\Omega - 2\psi_r + 2 \log 2) - \beta h \operatorname{Ein} 4\beta h - 2\beta h \operatorname{Ein} 2\beta h \\ &- j \operatorname{Ein} 4\beta h - 2 \sin 2\beta h - 2j(\cos 2\beta h - 1)].\end{aligned} \tag{35}$$

IV. The required equality of ψ_r and ψ_t. Although the method of King and Middleton yields a different expansion parameter in the receiving case from that found by them for the driven dipole, it is necessary, in order to have a consistent theory for the receiving dipole, that both expansion parameters be identical. This can be seen from a comparison of the two equivalent definitions for effective length. In addition to Eq. (30), effective length may be defined by

$$d = I_{0E} Z_a / E_i . \tag{36}$$

*The function $\operatorname{Ein}(x)$ is defined in the Appendix.

[7]S. H. Dike, *The effective length of antennas*, Tech. Report No. 13, Radiation Laboratory, Johns Hopkins University, 1951. (Submitted to I.R.E.).

From (30) and (36) one obtains the interesting equality that

$$\frac{I_{0E}}{E_i} = \frac{1}{V_0} \int_{-h}^{h} I_v(z)\, dz. \tag{37}$$

Substituting in (37) from (2), (14), and (29), one obtains

$$\frac{2f_E(0)}{\psi_r H_1} = \frac{\beta g_v(h)}{\psi_t H_2}. \tag{38}$$

This relation is true if $\psi_t = \psi_r$, because when this is so, $H_1 = H_2$ and $2f_E(0)$ does indeed equal $\beta g_v(h)$ for all orders of solution. A second relation is possible between ψ_r and ψ_t which satisfies (38), but this relation is dependent upon the order of the solution. This means that the expansion parameter would be a function of the number of terms retained in the solutions for the currents. This is obviously undesirable. In any case ψ_r must equal ψ_t for the zero order solution.

This fact constitutes the first difficulty encountered in the King-Middleton method. It is difficult to say which of the two parameters, ψ_t or ψ_r, is the better. It appears that neither is particularly good.[1]

V. The behavior of the theory for short dipoles. It was pointed out in Ref. 1 that the values of gain obtained from Eq. (26) do not reduce in the limit of decreasing βh to the value 1.5. One would expect the value 1.5 as being the correct one for finite Ω because for very small βh the current distribution must be essentially linear. This does not imply that either the radiation resistance or the effective length should reduce to those of the indefinitely thin short dipole. For very small βh, Eq. (26) becomes

$$G_t = \frac{3}{2}\left[\frac{(2\psi_t - \Omega + 3)^2}{3\psi_t^2 - 2\psi_t(\Omega - 2 - 2\log 2)}\right], \qquad \beta h \ll 1. \tag{39}$$

This result is independent of βh but remains a function of the expansion parameter. Since for small βh, $\psi_t = \Omega - 2$, Eq. (39) becomes

$$G_t = \frac{3}{2}\left[\frac{(\Omega - 1)^2}{(\Omega - 2)(\Omega - 2 + 4\log 2)}\right], \qquad \beta h \ll 1. \tag{40}$$

This is not 1.5 except for $\Omega \to \infty$. It is not evident that the difficulty would be removed by retaining more terms of the series solution. Various values of short dipole gain can be obtained from (39) by using the expansion parameters of previous authors. This is shown for $\Omega = 10$ in Table I, where the value of the expansion parameter for small βh is given.

TABLE I

Author	ψ	G_t	*Percent Error*
Hallén	Ω	1.5114	0.76
Gray[8]	$\Omega - 2 + \log 4$	1.4835	1.10
King-Middleton	$\Omega - 2$	1.4098	6.01

If the parameter ψ_r is used, where $\psi_r = \Omega - 1$ for $\beta h \ll 1$, the value of G_t at $\Omega = 10$ is 1.464 which is an improvement over the use of ψ_t.

[8] M. C. Gray, *A modification of Hallén's solution of the antenna problem*, J. Appl. Phys. 15, 61-65 (1944).

The fact that the bracket of (39) should be unity in the first-order theory gives

$$\psi^2 - \psi(2\Omega - 8 + 4 \log 2) + \Omega^2 - 6\Omega + 9 = 0,$$

or

$$\psi = \Omega - 2.6138 \pm 0.879\,(\Omega - 2.807)^{1/2}. \tag{41}$$

Of the two values allowed, the larger is probably the one that should be chosen.

A similar situation results in the value of σ/λ^2 for very short matched-loaded dipoles. The theory should reduce to the value $9/16\pi$. In the limit of decreasing βh, Eq. (33) for the matched-loaded case becomes

$$\frac{\sigma}{\lambda^2} = \frac{9}{16\pi}\left[\frac{\psi_t(2\psi_t - \Omega + 3)(2\psi_r - \Omega + 3)}{\psi_r^2(3\psi_t - 2\Omega + 4 + 4 \log 2)}\right]^2. \tag{42}$$

If the requirement is made that $\psi_t = \psi_r$ then

$$\frac{\sigma}{\lambda^2} = \frac{9}{16\pi}\left[\frac{(2\psi - \Omega + 3)^2}{3\psi^2 - 2\psi(\Omega - 2 - 2 \log 2)}\right]. \tag{43}$$

The bracket of (43) is identical to that of (39) leading to the same requirement on ψ given by (41). It appears then that an additional requirement should be imposed on the expansion parameter that has not previously been considered.

It is also of interest to consider the value of impedance in the limit of very small βh. The impedance is given by

$$Z_a = V_0/I_{0s} = -j\zeta\psi_t H_2[2\pi f_s(0)]^{-1}, \tag{44}$$

which for $\beta h \ll 1$, becomes

$$Z_a = R_a + jX_a = \frac{-60\psi_t[3\psi_t + 2j(\beta h)^3]}{3\beta h(2\psi_t + x) + j(\beta h)^4}, \tag{45}$$

where

$$x = 2 + 2 \log 2 - \Omega. \tag{46}$$

Separating real and imaginary parts of (45):

$$R_a = 20\psi_t(\beta h)^2(3\psi_t + 2x)(2\psi_t + x)^{-2}, \tag{47}$$

and

$$X_a = -\frac{60\psi_t^2}{\beta h(2\psi_t + x)}. \tag{48}$$

Since $\psi_t = \Omega - 2$ for $\beta h \ll 1$,

$$R_a = \frac{20(\beta h)^2}{1 + (2 \log 2)^2/y}, \tag{49}$$

where

$$y = \Omega^2 + (\log 2 - 1)(\log 2 - 1 + 4\Omega), \tag{50}$$

and

$$X_a = \frac{-60}{\beta h}\left[\frac{(\Omega - 2)^2}{\Omega - 2 + 2 \log 2}\right]. \tag{51}$$

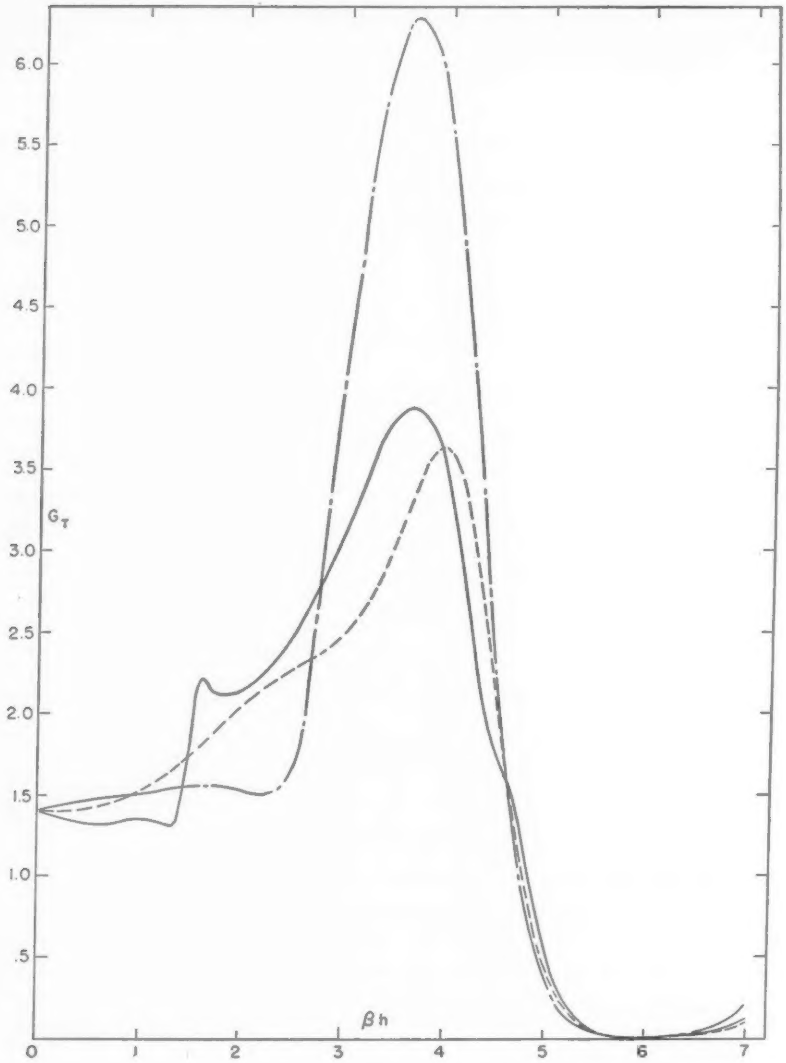

FIG. 6. Absorption Gain for $\Omega = 10$

——— Equation (25) using second-order $|Z_a|^2/R_a$

-------- First-order theory

— · — Equation (27) using second-order R_a

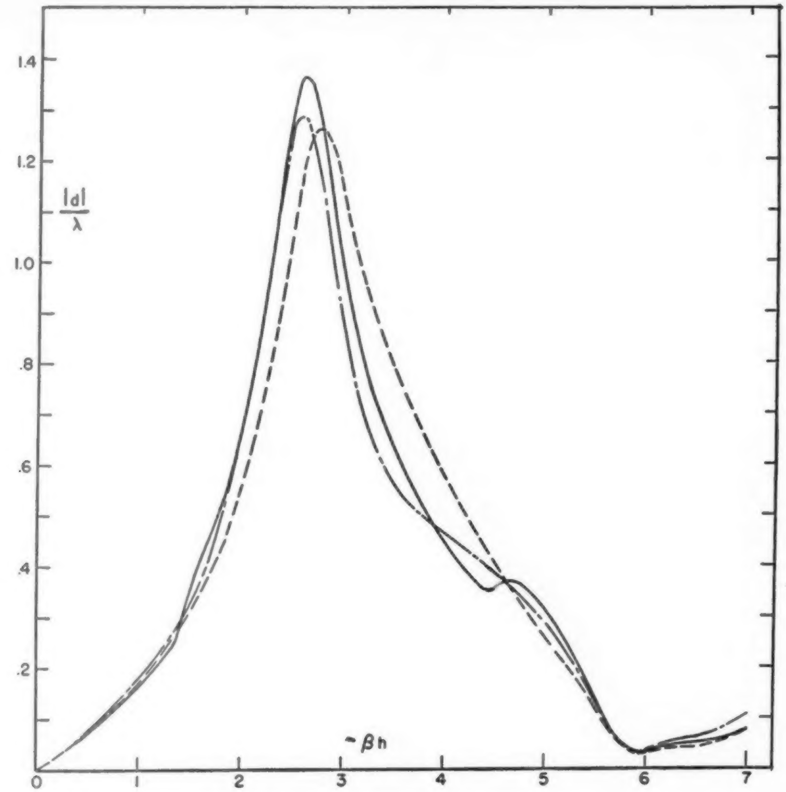

FIG. 7. Effective length for $\Omega = 10$

-------- First-order theory
— · — Equation (27) using second-order R_a
——— Equation (32) using second-order Z_a

For a value of ψ which satisfies (41), R_a becomes

$$R_a = 20(\beta h)^2(2\psi - \Omega + 3)^2(2\psi - \Omega + 2 + 2 \log 2)^{-2}. \tag{52}$$

Using ψ from (41) for $\Omega = 10$, $R_a = 20(\beta h)^2(0.941)$. The corresponding value from the King-Middleton relation (49) is $R_a = 20(\beta h)^2(0.978)$. There is a difference of about four percent between these two values.

The corresponding values of X_a for $\Omega = 10$ are $X_a = -(7.375)60/\beta h$ for ψ satisfying (41), and $X_a = -(6.819)60/\beta h$ for the King-Middleton expression (51). The difference here is about eight percent.

It is of interest to compare these results with the reactance computed from the static capacitance between two cylinders placed end to end in air, and separated by a distance

which is negligible compared with their individual lengths. This reactance is given by[9]

$$X_{as} = \frac{-\zeta(\Omega - \log 12)}{2\pi\beta h} = \frac{-60(\Omega - 2.485)}{\beta h}. \tag{53}$$

For $\Omega = 10$, Eq. (53) gives $X_{as} = -(7.515)60/\beta h$. This result may be compared with those obtained from Eq. (48) listed in Table II.

TABLE II

ψ	$\frac{-\beta h X_a}{60}$ for $\Omega = 10$, $\beta h \ll 1$
Ω (Hallén)	7.470
from Eq. (41)	7.375
$\Omega - 2 + \log 4$ (Gray)	7.246
ψ_r	7.114
$\Omega - 2$ (King-Middleton)	6.819

VI. The use of the King-Middleton values of impedance. It was thought that improved values of the various antenna properties might be obtained by using the second-order impedance values of King and Middleton in the expressions for gain, back-scattering cross section, and effective length. Figure 6 shows the absorption gain, G_t, computed from the straight first-order theory, and also as computed from Eq. (25) using second-order values of $|Z_a|^2/R_a$, and from Eq. (27) using first-order effective length and second-order R_a. As can be seen from this figure, very different results are obtained when the King-Middleton values are used. The use of Eq. (27) with second-order R_a would seem to indicate that the King-Middleton values for the radiation resistance are too large in the region of $\beta h = 2$, and are too small in the region of $\beta h = 3.5$. The use of Eq. (25) with second-order $|Z_a|^2/R_a$ yields results which are certainly contrary to fact near $\beta h = 1.5$ and $\beta h = 4.5$. The "bulge" in the first-order theory near $\beta h = 2$ is contrary to experimental data and disappears if ψ_r is used instead of ψ_t.[1]

Figure 7 is a comparison of the first-order theory for effective length given by Eq. (31), the relation of Eq. (32) using second-order Z_a, and Eq. (27) using second-order R_a and first-order G_t. The use of (32) with second-order Z_a gives results which are unreasonable near $\beta h = 1.5$ and $\beta h = 4.5$.

Figure 8 shows the back-scattering cross section for matched load according to the first-order theory and also from Eq. (33) where second-order King-Middleton values are used for Z_a. The latter curve behaves strangely near $\beta h = 1.5$. Neither curve represents experiment, particularly in the region above $\beta h = 4$.[1]

These three figures show that it is not permissible to use second-order King-Middleton values of impedance in first-order formulas. This may be due to the unknown behavior of the series solution as regards convergence,[10] or it may be that the second-order impedance values of King and Middleton are not good. This latter case would imply that the expansion parameter ψ_t can be better chosen.

VII. The problem of choosing the expansion parameter. The integral equation of

[9] R. W. P. King and C. Harrison, Jr., *The impedance of short, long and capacity loaded antennas with a critical discussion of the antenna problem*, J. Appl. Phys., **17**, 170 (1944).

[10] S. Schelkunoff, *Concerning Hallén's integral equation for cylindrical antennas*, Proc. I.R.E., **33**, 872 (1945).

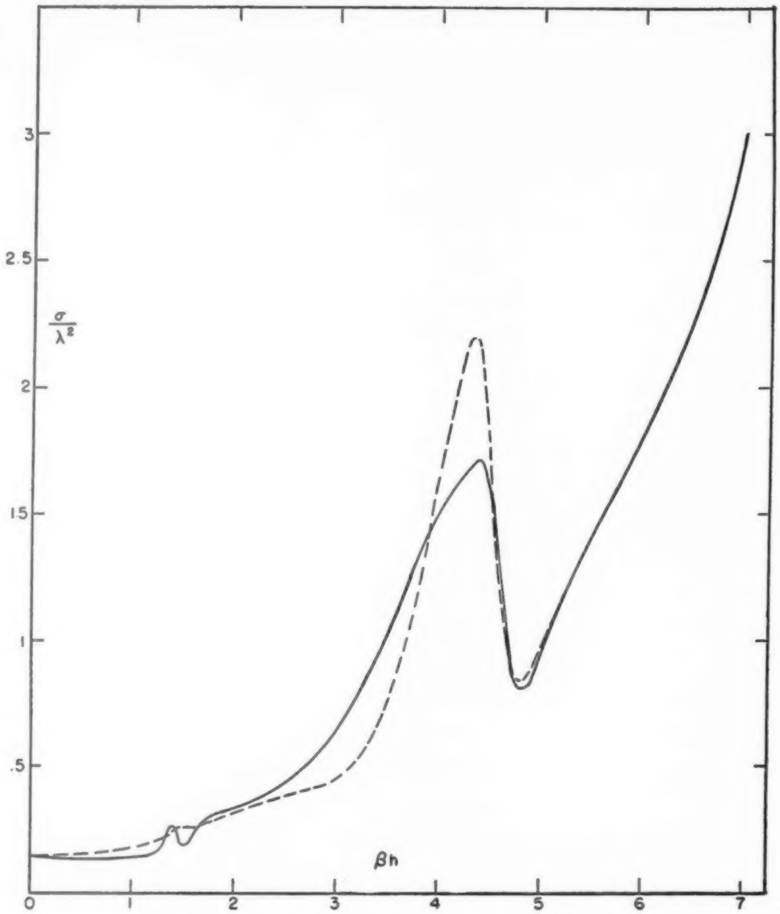

FIG. 8. Back-scattering cross-section for $\Omega = 10$, $Z_L = Z_a^*$

-------- First-order theory

——— Equation (33) using second-order $|z_a|^2 R_a$

Hallén is known to be a sufficiently accurate formulation of the problem. It has been examined by many workers,[11,12,13] and has been shown to contain approximations only to the order $(a/h)^2$.[10] Hallén proposed an iterative process for solving this equation and obtained a series solution.[2] Modified solutions have been proposed by Miss Gray[8] and by King and Middleton.[3] These have consisted essentially of modifying the expansion

[11] D. Middleton and R. W. P. King, *The thin cylindrical antenna: a comparison of theories*, J. Appl. Phys. **17**, 273-284 (1946).

[12] L. Brillouin, *The antenna problem*, Q. Appl. Math. **1**, 201-214 (1943).

[13] S. Schelkunoff, *Antenna theory and experiment*, J. Appl. Phys. **15**, 54-60 (1944).

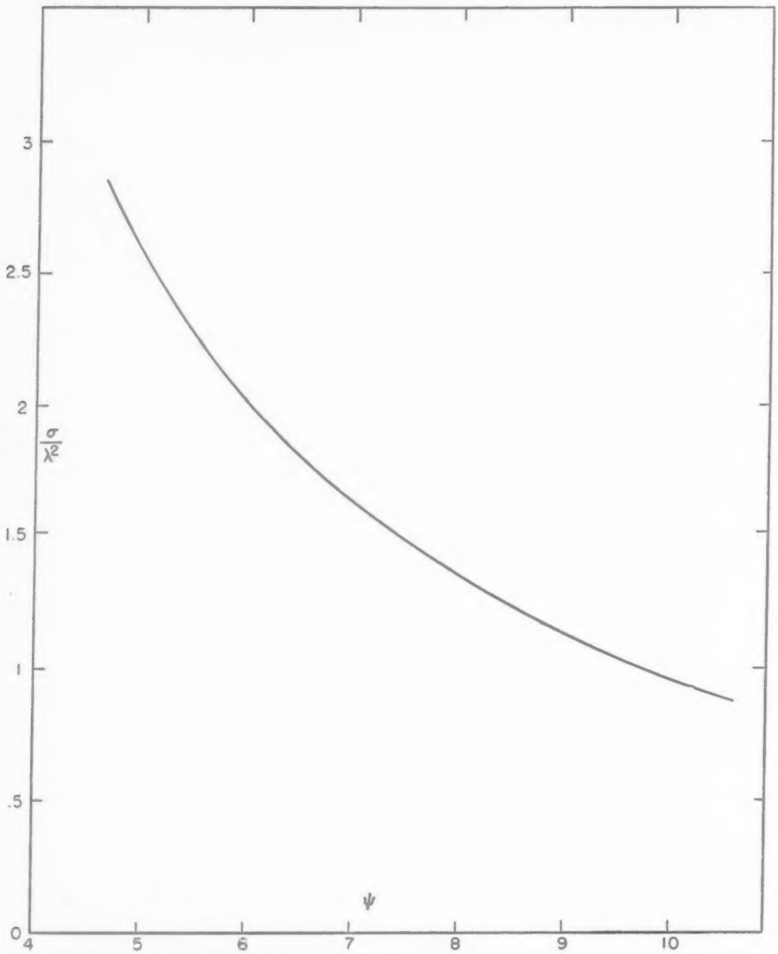

FIG. 9. Back-scattering cross-section at $\beta h = 2\pi$ for $\Omega = 10$, $Z_L = 0$, as a function of expansion parameter

parameter used. Hallén[14] claims that such modifications have no mathematical foundation, and this may be so. However, the method of King and Middleton does appear reasonable from the standpoint of physical reasoning. The choice of a trial function which is known to be representative of the current distribution in the limit of vanishing dipole radius seems to have some merit, although the arguments as to why this should yield a better solution have been attacked by Hallén.[15] Nevertheless, if it is thought

[14] E. Hallén, *Admittance diagrams for antennas and the relation between antenna theories*, Tech. Report No. 46, Cruft Laboratory, Harvard Univ., 1948.

[15] E. Hallén, *Traveling waves and unsymmetrically fed antennas*, Tech. Report No. 49, Cruft Laboratory, Harvard Univ., 1948.

that such a choice of a trial function is justified, it appears from the foregoing discussion that consideration must be given to *both* Eqs. (11) and (19) in making a final choice of ψ. Note that these two expressions are identical at all resonant lengths ($\beta h = \pi/2$, $3\pi/2$, $5\pi/2$, $\cdots$). Also since $\Psi(z)$ is not predominantly real for the longer lengths, the final choice of ψ may best be a complex value.

Back-scattering cross section appears to be a property of the dipole antenna which is particularly sensitive to prediction by theory. As an illustration of its sensitivity to the choice of expansion parameter, Fig. 9 shows the back-scattering cross section at $\beta h = 2\pi$, $\Omega = 10$, for the shorted dipole as a function of the expansion parameter. A factor greater than two exists between the result using the King-Middleton parameter of about 6 at this length, and the Hallén parameter of 10 in the first-order theory.

VIII. Conclusion. In view of the fact that the series solution of the integral equation has been studied, criticized, and modified by many authors since Hallén's first paper in 1938, and since a theory which can be practically computed does not seem to exist which adequately predicts the complete behavior of a simple dipole antenna, it appears perhaps that a new attack on the problem is justified.

It is significant that the results of Van Vleck, et al,[16] for the back-scattering cross section of a shorted dipole agree more closely with experiment than the first-order solutions of Hallén, King and Middleton, or Miss Gray. Such a comparison is made in Fig. 16 of Ref. 1. Hallén's recent solution[15] for the driven dipole may be an improved one from the standpoint of gain. Some attempts have been made to solve the integral equation by variational methods.[17,18] Storer's solution fails for βh greater than $3\pi/2$. Tai removed this difficulty but his first-order values of R_a at the first resonant length are still higher than those of King and Middleton. It may be worthwhile to follow up a suggestion made by Brillouin[19] that the known function and the kernel of the integral equation be expanded in Fourier series with known coefficients, and that the unknown function for the current be likewise expanded with unknown coefficients. Term-by-term integration would then lead to a set of simultaneous equations for determination of the coefficients. No published results of such an approach have come to the author's attention.

APPENDIX

$$\mathrm{Ein}\,(x) = \mathrm{Cin}\,(x) + j\,\mathrm{Si}\,(x)$$

$$\mathrm{Cin}\,(x) = \int_0^x \frac{1 - \cos t}{t}\,dt$$

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t}\,dt$$

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[16] J. H. Van Vleck, F. Bloch and M. Hamermesh, *Theory of radar reflection from wires or thin metallic strips*, J. Appl. Phys. **18**, 274 (1947).

[17] J. E. Storer, *Variational solution to the problem of the symmetrical cylindrical antenna*, Tech. Report No. 101, Cruft Laboratory, Harvard Univ., 1950.

[18] C. T. Tai, *A variational solution to the problem of cylindrical antennas*, Tech. Report No. 12, SRI Project No. 188, Stanford Research Institute, 1950.

[19] L. Brillouin, *Antennas for ultra-high frequencies*, Electrical Communication **22**, 11 (1944).

ON BROWNIAN MOTION, BOLTZMANN'S EQUATION, AND THE FOKKER-PLANCK EQUATION*

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Abstract. In order to describe Brownian motion rigorously, Boltzmann's integral equation must be used. The Fokker-Planck type of equation is only an approximation to the Boltzmann equation and its domain of validity is worth examining.

A treatment of the Brownian motion in velocity space of a particle with known initial velocity based on Boltzmann's integral equation is given. The integral equation, which employs a suitable scattering kernel, is solved and its solution compared with that of the corresponding Fokker-Planck equation. It is seen that when M/m, the mass ratio of the particles involved, is sufficiently high and the dispersion of the velocity distribution sufficiently great, the Fokker-Planck equation is an excellent description. Even when the dispersion is small, the first and second moments of the Fokker-Planck solution are reliable. The higher moments, however, are then in considerable error—an error which becomes negligible as the dispersion increases.

1. In the treatment of Brownian motion, it is customary to assume a Langevin equation and simple dynamical statistics of the individual collisions and then to deduce a Fokker-Planck equation describing the random motion of the heavy particle. The Fokker-Planck equation obtained is a second-order partial differential equation and the absence of higher-order differential terms is inferred directly from the above assumptions. As will be seen, the solution of this Fokker-Planck equation does not provide a completely satisfactory physical description. Consequently, the assumptions underlying the equation cannot be correct [1, 2, 3, 4] and the extent of their approximate validity comes under question.

That the solution of the Fokker-Planck equation is not a wholly satisfactory representation of Brownian motion may be seen in the following way. Consider a heavy particle known to have the velocity $\mathbf{v}_0$ at $t = 0$. For all subsequent time, there is a finite probability that the particle will have undergone no collision. It must, therefore, be expected that the probability density $w(\mathbf{v}, t)$** describing the stochastic motion in velocity space will always have a singular component of the form $f(t)\delta(\mathbf{v} - \mathbf{v}_0)$, where $\delta(\mathbf{v} - \mathbf{v}_0)$ is the Dirac delta-function. If one were to try to describe the motion by the Fokker-Planck equation

$$\frac{\partial w(\mathbf{v}, t)}{\partial t} = \frac{1}{2} D\nabla^2 w(\mathbf{v}, t) + \eta\nabla\cdot\{\mathbf{v}w(\mathbf{v}, t)\}, \tag{1}$$

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**The function $w(\mathbf{v}, 0)$ obeying (2) is often represented in the literature by $P_2(\mathbf{v}_0/\mathbf{v}; t)$, the probability density for velocity $\mathbf{v}$, t seconds after there is a known velocity $\mathbf{v}$.

subject to the initial condition

$$w(\mathbf{v}, 0) = \delta(\mathbf{v} - \mathbf{v}_0), \tag{2}$$

no such singular component would be available in the solution. The immediate disappearance of an initial singularity is, indeed, characteristic of all diffusion equations of finite order. Only by means of an integral equation can such a singularity be maintained.

All of this, of course, is in keeping with the fact that fundamental to the description of Brownian motion is Boltzmann's equation, an integral equation of the type desired [4]. If $A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}$ is the probability per unit time that a particle with velocity $\mathbf{v}'$ will undergo a transition to a volume $d\mathbf{v}$ about $\mathbf{v}$, the Boltzmann equation describing the motion is

$$\frac{\partial w(\mathbf{v}, t)}{\partial t} = \int w(\mathbf{v}', t) A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}' - w(\mathbf{v}, t) \int A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}'. \tag{3}$$

This is simply an expression of the fact that the rate of change of the population of a cell in velocity space is the difference between the rate of departures from the cell and the rate of arrivals.

From this Boltzmann equation a corresponding Fokker-Planck equation may be derived. If Eq. (3) is multiplied by an arbitrary, but suitably behaved function $R(\mathbf{v})$, and integrated over $\mathbf{v}$,

$$\begin{aligned}
\int R(\mathbf{v}) \frac{\partial w(\mathbf{v}, t)}{\partial t}\, d\mathbf{v} &= \iint R(\mathbf{v}) w(\mathbf{v}', t) A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}'\, d\mathbf{v} \\
&\qquad - \iint R(\mathbf{v}) w(\mathbf{v}, t) A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}\, d\mathbf{v}' \\
&= \iint \left\{ \sum_0^\infty \frac{(\mathbf{v} - \mathbf{v}')^{(n)}}{n!} \cdot \nabla'^{(n)} R(\mathbf{v}') \right\} w(\mathbf{v}', t) A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}\, d\mathbf{v}' \qquad (4) \\
&\qquad - \iint R(\mathbf{v}) w(\mathbf{v}, t) A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}\, d\mathbf{v}' \\
&= \iint \left\{ \sum_1^\infty \frac{(\mathbf{v} - \mathbf{v}')^{(n)}}{n!} \cdot \nabla'^{(n)} R(\mathbf{v}') \right\} w(\mathbf{v}', t) A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}'\, d\mathbf{v}.
\end{aligned}$$

Here $(\mathbf{v} - \mathbf{v}')^{(n)} \cdot \nabla'^{(n)}$ is to be understood as the dot product of two n-th-rank tensors. Integrating by parts, one has

$$\int R(\mathbf{v}') \frac{\partial w(\mathbf{v}', t)}{\partial t}\, d\mathbf{v}' = \iint R(\mathbf{v}') \sum_0^\infty \frac{1}{n!} \nabla'^{(n)} \cdot \{(\mathbf{v}' - \mathbf{v})^{(n)} A(\mathbf{v}', \mathbf{v}) W(\mathbf{v}', t)\}\, d\mathbf{v}'\, d\mathbf{v}. \tag{5}$$

Since $R(\mathbf{v}')$ is an arbitrary function, the associated coefficients may be equated to yield

$$\frac{\partial w(\mathbf{v}, t)}{\partial t} = \sum_1^\infty \frac{1}{n!} \nabla^{(n)} \cdot \{ A_n(\mathbf{v}) w(\mathbf{v}, t) \}, \tag{6}$$

where $A_n(\mathbf{v})$ is the tensor

$$A_n(\mathbf{v}) = \int (\mathbf{v} - \mathbf{v}')^{(n)} A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}'. \tag{7}$$

Equations (3) and (6) are equivalent and provide an *exact* description of Brownian motion.

This treatment may be readily generalized to include Brownian motion in coordinate and velocity space.

2. In those treatments of Brownian motion based on Langevin's equation, moments higher than the second are found to vanish, and the Fokker-Planck equation (3) is obtained. As already noted, such an equation is certainly unsatisfactory when the dispersion is small. It would, therefore, be desirable to try to treat the Boltzmann equation directly. Plainly an exact kernel $A(\mathbf{v}, \mathbf{v}')$ is unavailable and its use is almost certainly not feasible. However, it is possible to introduce a kernel which provides a reasonably accurate description of the microscopic scattering process and which is, at the same time, amenable to treatment. Such a kernel is of the form $A(\mathbf{v}, \mathbf{v}') = \mathcal{A}(\mathbf{v}' - \gamma\mathbf{v})$, where γ is a dynamical damping parameter close in value to, but less than, one. Some justification for this form may be found along the following lines:

Let $B(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}'$ be the probability per unit time of a particle with initial velocity v making a transition to a volume element $d\mathbf{v}'$ about $\mathbf{v}'$, when all the particles with which the heavy particle collides are stationary. If the lighter particles have an equilibrium distribution $w(\mathbf{v}'')$, then

$$A(\mathbf{v}, \mathbf{v}') = \int B(\mathbf{v} - \mathbf{v}'', \mathbf{v}' - \mathbf{v}'')w(\mathbf{v}'')\, d\mathbf{v}''. \tag{8}$$

Since the particle under observation is very much heavier than the particles with which it collides, $B(\mathbf{v}, \mathbf{v}')$ is a highly localized function of $\mathbf{v}'$, centered roughly about $\gamma\mathbf{v}$ where again γ is very close to but less than unity. If $B(\mathbf{v}, \mathbf{v}')$ is assumed to have the form $B(\mathbf{v}' - \gamma\mathbf{v})$, then

$$\begin{aligned} A(\mathbf{v}, \mathbf{v}') &= \int B[\mathbf{v}' - \mathbf{v}'' - \gamma(\mathbf{v} - \mathbf{v}'')]w(\mathbf{v}'')\, d\mathbf{v}'' \\ &= \int B[\mathbf{v}' - \gamma\mathbf{v} - (1 - \gamma)\, \mathbf{v}'']w(\mathbf{v}'')\, d\mathbf{v}'', \end{aligned}$$

so that this will have the form of $\mathcal{A}(\mathbf{v}' - \gamma\mathbf{v})$.

Note that the form of $\mathcal{A}(\mathbf{v}' - \gamma\mathbf{v})$ implies that the mean free time τ of a heavy particle is independent of its velocity, since

$$\frac{1}{\tau(\mathbf{v})} = \int A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}' = \int \mathcal{A}(\mathbf{v}' - \gamma\mathbf{v})\, d\mathbf{v}' = \int \mathcal{A}(\mathbf{v}'')\, d\mathbf{v}'', \tag{9}$$

a constant. This behavior is proper to Brownian motion where the heavy particle moves so slowly compared to the lighter particles that the mean relative velocity of the heavy particle does not vary significantly.

It would appear offhand that the functional form of $\mathcal{A}(\mathbf{v})$ could be chosen arbitrarily. However, this is not the case since $\mathcal{A}(\mathbf{v}, \mathbf{v}')$ must satisfy the equilibrium condition:

$$\omega(\mathbf{v}')A(\mathbf{v}', \mathbf{v}) = \omega(\mathbf{v})A(\mathbf{v}, \mathbf{v}') \tag{10}$$

where $\omega(\mathbf{v})$ is the equilibrium distribution of the heavy particles which the particle will ultimately assume. If it is also demanded that $\omega(\mathbf{v})$ depend only on $|\mathbf{v}|$, the two restrictions imply that $\mathcal{A}(\mathbf{v})$ must have the form $\mathcal{A}_0 \exp\{-\beta\mathbf{v}^2\}$ and that $\omega(\mathbf{v})$ must have the corresponding form $\omega_0 \exp\{-\beta(1 - \gamma^2)\mathbf{v}^2\}$, where $\mathcal{A}_0$ and ω_0 are constants (see Appendix 1). That the Gaussian character of the equilibrium distribution follows from the form of $\mathcal{A}(\mathbf{v} - \gamma\mathbf{v}')$ is reassuring.

Thus, the Boltzmann equation to be solved is

$$\frac{\partial w(\mathbf{v}, t)}{\partial t} = \mathcal{A}_0 \int w(\mathbf{v}', t) \exp \{-\beta(\mathbf{v} - \gamma\mathbf{v}')^2\} \, d\mathbf{v}' - \frac{1}{\tau} w(\mathbf{v}, t), \tag{11}$$

where

$$\begin{aligned} \frac{1}{\tau} = \int \mathcal{A}(\mathbf{v}' - \gamma\mathbf{v}) \, d\mathbf{v}' &= \mathcal{A}_0 \int \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathcal{A}_0 \int \exp \{-\beta\mathbf{v}''^2\} \, d\mathbf{v}'' = \mathcal{A}_0 \left(\frac{\pi}{\beta}\right)^{3/2}. \end{aligned} \tag{12}$$

Before discussing the solution of this equation, it is worth while to put down the corresponding Fokker-Planck equation. The first moment will be given by

$$\begin{aligned} A_1 &= \mathcal{A}_0 \int (\mathbf{v} - \mathbf{v}') \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathcal{A}_0 \int [(\gamma\mathbf{v} - \mathbf{v}') + (1 - \gamma)\mathbf{v}] \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathcal{A}_0(1 - \gamma)\mathbf{v} \int \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathcal{A}_0\left(\frac{\pi}{\beta}\right)^{3/2}(1 - \gamma)\mathbf{v} = \left(\frac{1 - \gamma}{\tau}\right)\mathbf{v}. \end{aligned} \tag{13}$$

Similarly for the second moment,

$$\begin{aligned} A_2 &= \mathcal{A}_0 \int (\mathbf{v} - \mathbf{v}')(\mathbf{v} - \mathbf{v}') \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathcal{A}_0 \int [(\mathbf{v}' - \gamma\mathbf{v})(\mathbf{v}' - \gamma\mathbf{v}) + (1 - \gamma)^2\mathbf{v}\mathbf{v}] \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \frac{\mathcal{A}_0}{2} \pi^{3/2}\beta^{-5/2}\epsilon + (1 - \gamma)^2\mathbf{v}\mathbf{v}\mathcal{A}_0\left(\frac{\pi}{\beta}\right)^{3/2} \\ &= \frac{1}{2\beta\tau} \epsilon + \frac{(1 - \gamma)^2\mathbf{v}\mathbf{v}}{\beta\tau}, \end{aligned} \tag{14}$$

where ϵ is the unit tensor.

If the latter part of A_2 is ignored since $(1 - \gamma)^2$ is small and if the higher moments (whose effect will be small for $t \gg \tau$) are ignored, Eq. (1) is regained where now

$$D = \frac{1}{2\beta\tau} \quad \text{and} \quad \eta = \frac{1 - \gamma}{\tau}. \tag{15}$$

As is seen in Appendix 2, the solution of the Boltzmann equation (11) subject to condition (2) is given by

$$w_B(\mathbf{v}, t) = \left[\delta(\mathbf{v} - \mathbf{v}_0) + \sum_1^\infty \frac{1}{n!} \left(\frac{t}{\tau}\right)^n \left(\frac{\beta}{\pi\Delta_n}\right)^{3/2} \exp \left\{-\frac{\beta}{\Delta_n}(\mathbf{v} - \gamma^n\mathbf{v}_0)^2\right\}\right] \exp \left\{-\frac{t}{\tau}\right\}, \tag{16}$$

where

$$\Delta_n = \frac{1 - \gamma^{2n}}{1 - \gamma^2}. \tag{17}$$

The solution of the Fokker-Planck equation (1) subject to condition (2) may be taken directly from Wang and Uhlenbeck [1] and is given by

$$w_{FP}(\mathbf{v}, t) = \left[\frac{\eta}{\pi D(1 - \exp\{-2\eta t\})}\right]^{3/2} \exp\left\{\frac{-\eta(\mathbf{v} - \mathbf{v}_0 \exp\{-\eta t\})^2}{D(1 - \exp\{-2\eta t\})}\right\}. \tag{18}$$

Note that the singularity $\delta(\mathbf{v} - \mathbf{v}_0)$ is preserved in the solution of the integral equation but is not in the solution of the Fokker-Planck equation. For $t \gg \tau$, however, the delta-function ceases to play a prominent role.

From Eqs. (16), (17) one finds that the equilibrium distribution for the solution of the Boltzmann equation is given by

$$\omega_B(\mathbf{v}) = \lim_{t\to\infty} w_B(\mathbf{v}, t) = \left[\frac{\beta(1 - \gamma^2)}{\pi}\right]^{3/2} \exp\{-\beta(1 - \gamma^2)v^2\}. \tag{19}$$

For the Fokker-Planck equation,

$$\omega_{FP}(\mathbf{v}) = \left(\frac{\eta}{\pi D}\right)^{3/2} \exp\left\{-\frac{\eta}{D}v^2\right\}. \tag{20}$$

Inserting the values of η, D from (15), this becomes:

$$\omega_{FP}(\mathbf{v}) = \left(\frac{2(1 - \gamma)\beta}{\pi}\right)^{3/2} \exp\{-2\beta(1 - \gamma)v^2\}. \tag{21}$$

Plainly if γ is sufficiently close to one, then

$$(1 - \gamma^2) = (1 - \gamma)(1 + \gamma) \simeq 2(1 - \gamma)$$

and the two equilibrium distributions are identical.

It is also of interest to compare the manner in which the average velocity and the variance vary in time. These quantities are defined by

$$\langle\mathbf{v}\rangle(t) = \int \mathbf{v}w(\mathbf{v}, t)\, d\mathbf{v}$$

and

$$\sigma^2(t) = \int (v^2 - \langle\mathbf{v}\rangle^2)w(\mathbf{v}, t)\, d\mathbf{v}.$$

$\langle\mathbf{v}\rangle_B(t)$, $\langle\mathbf{v}\rangle_{FP}(t)$, $\sigma_B^2(t)$, and $\sigma_{FP}^2(t)$ may be computed directly from their respective equations of motion. Thus, if Eq. (1) is multiplied on both sides by $\mathbf{v}$ and integrated over $\mathbf{v}$, then

$$\frac{d\langle\mathbf{v}\rangle}{dt} = -\eta\langle\mathbf{v}\rangle, \qquad \text{so that} \qquad \langle\mathbf{v}\rangle_{FP}(t) = \mathbf{v}_0 \exp\{-\eta t\} = \mathbf{v}_0 \exp\left\{-\frac{1 - \gamma}{\tau}t\right\}. \tag{22}$$

Similarly, multiplying by $\mathbf{v}^2$ and integrating, it is found that

$$\frac{d\langle\mathbf{v}^2\rangle}{dt} = 3D - 2\eta\langle\mathbf{v}^2\rangle, \tag{23}$$

which yields in turn

$$\frac{d\sigma^2}{dt} = \frac{d}{dt}(\langle \mathbf{v}^2 \rangle - \langle \mathbf{v} \rangle^2) = 3D - 2\eta\sigma^2; \tag{24}$$

and so

$$\sigma_{FP}^2(t) = \frac{3D}{2\eta}(1 - \exp\{-2\eta t\}) = \frac{3}{4\beta(1-\gamma)}\left(1 - \exp\left\{-\frac{2}{\tau}(1-\gamma)t\right\}\right). \tag{25}$$

The same procedure may be applied to the integral equation (11) to give

$$\langle \mathbf{v} \rangle_B(t) = \mathbf{v}_0 \exp\left\{-\left(\frac{1-\gamma}{\tau}\right)t\right\} \tag{26}$$

and

$$\langle \mathbf{v}^2 \rangle_B(t) = \frac{3}{2\beta(1-\gamma^2)}\left[1 - \exp\left\{-\left(\frac{1-\gamma^2}{\tau}\right)t\right\}\right] + \mathbf{v}_0^2 \exp\left\{-\left(\frac{1-\gamma^2}{\tau}\right)t\right\}. \tag{27}$$

Correspondingly, one finds that

$$\begin{aligned}\sigma_B^2(t) = \frac{3}{2\beta(1-\gamma^2)}&\left[1 - \exp\left\{-\frac{(1-\gamma^2)t}{\tau}\right\}\right] \\ &+ \mathbf{v}_0^2\left[\exp\left\{-\left(\frac{1-\gamma^2}{\tau}\right)t\right\} - \exp\left\{-2\left(\frac{1-\gamma}{\tau}\right)t\right\}\right].\end{aligned} \tag{28}$$

These same results could also have been obtained from the solutions (16) and (18), but the computations are more tedious.

It is seen that $\langle \mathbf{v}_B \rangle(t)$ and $\langle \mathbf{v}_{FP} \rangle(t)$ are identical and that $\sigma_B^2(t)$ and $\sigma_{FP}^2(t)$ are nearly identical. Indeed, if the smaller term in A_2 had not been ignored in obtaining the corresponding Fokker-Planck approximation, $\sigma_B^2(t)$ and $\sigma_{FP}^2(t)$ would have been precisely the same. For consider the Boltzmann equation in its differential form:

$$\frac{\partial w}{\partial t} = \sum_1^\infty \nabla^{(n)} \cdot (A_n w).$$

The above procedure yields

$$\frac{d\langle \mathbf{v}^2 \rangle}{dt} = \int \mathbf{v}^2 \nabla^{(1)} \cdot (A_1 w)\, d\mathbf{v} + \int \mathbf{v}^2 \nabla^{(2)} \cdot (A_2 w)\, d\mathbf{v},$$

since all integrals involving higher moments vanish when integration by parts is carried out. Moreover, from the choice of $A(\mathbf{v}, \mathbf{v}')$, the two integrals are simple functions of $\langle \mathbf{v}^2 \rangle$ and the above differential equation does determine $\langle \mathbf{v}^2 \rangle(t)$. The same procedure applied to the Fokker-Planck equation can only yield the same result, because all contributing terms are present.

It is seen then that the validity of the Fokker-Planck approximation is excellent when γ is sufficiently close to one. For the ordinary domain of Brownian motion this will certainly be the case. For the elastic collision of hard spheres, for example, it is easily found that

$$\langle \delta \mathbf{v} \rangle = \frac{-4}{3}\frac{m}{M+m}\mathbf{v},$$

where $\langle \delta \mathbf{v} \rangle$ is the mean change in velocity suffered by a particle of mass M and velocity v in a single collision with particles of mass m. Then

$$\frac{d\langle \mathbf{v} \rangle}{dt} \simeq \frac{\langle \delta \mathbf{v} \rangle}{\tau} = \frac{-4}{3} \frac{m}{(M+m)\tau} \mathbf{v},$$

so that, from Eqs. (15) and (22)

$$\eta = \frac{(1-\gamma)}{\tau} \simeq \frac{4}{3} \frac{m}{(M+m)\tau}.$$

$1 - \gamma$ then is given by $4/3\ m/(M + m)$ and for typical Brownian motion will be extremely small.

If it were possible to treat the exact kernel $A(\mathbf{v}, \mathbf{v}')$, one would still expect to find excellent agreement between the Fokker-Planck and Boltzmann solutions for $t \gg \tau$. Even when $t \sim \tau$, the first and second moments of the Fokker-Planck equation should be reliable. But for $t \sim \tau$, higher-order moments would be in considerable error. However, for $t \gg \tau$, these errors will become entirely negligible.

References

1. M. C. Wang and G. E. Uhlenbeck, *On the theory of the Brownian Motion II*, Rev. Mod. Phys. **17**, 323 (1945).
2. S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
3. Lawson and Uhlenbeck, *Threshold signals*, (MIT) Radiation Laboratory Series, **24**, Chap. III, McGraw-Hill (1950).
4. J. Keilson, *The statistical nature of inverse Brownian Motion in velocity space*, Technical Report No. 127, Cruft Laboratory, Harvard University, May 10, 1951.

Appendix 1

The Restrictions on $\mathfrak{A}(\mathbf{v})$ and $\omega(\mathbf{v})$ Imposed by Equilibrium

Denoting the rectangular components of $\mathbf{v}$ by v_i, $i = 1, 2, 3$, and letting

$$\psi(v_1, v_2, v_3) = \ln \mathfrak{A}(\mathbf{v}),$$

the equilibrium relation (10) can be written in the form

$$\begin{aligned} \ln \omega(\mathbf{v}') + \psi(v_1 - \gamma v_1', v_2 - \gamma v_2', v_3 - \gamma v_3') \\ = \ln \omega(\mathbf{v}) + \psi(v_1' - \gamma v_1, v_2' - \gamma v_2, v_3' - \gamma v_3). \end{aligned} \tag{1-1}$$

Taking the partial derivative of Eq. (1-1) with respect to v_i, v_j', and noting that

$$\frac{\partial^2}{\partial v_i\, \partial v_j'} \ln \omega(\mathbf{v}') = \frac{\partial^2}{\partial v_i\, \partial v_j'} \ln \omega(\mathbf{v}) = 0,$$

it is seen that

$$-\gamma \psi_{ij}(v_1 - \gamma v_1', v_2 - \gamma v_2', v_3 - \gamma v_3') = -\gamma \psi_{ij}(v_1' - \gamma v_1, v_2' - \gamma v_2, v_3' - \gamma v_3), \tag{1-2}$$

where

$$\psi_{ij} = \frac{\partial^2}{\partial v_i \, \partial v_j} \psi(v_1 , v_2 , v_3).$$

Setting $v_i' = \gamma v_i$, $v_i'' = (1 - \gamma^2)v_i$ in (1–2), it is further observed that

$$\psi_{ij}(v_1'', v_2'', v_3'') = \psi_{ij}(0, 0, 0) = -\beta_{ij} ,$$

where β_{ij} is a constant. Hence $\psi(v_1 , v_2 , v_3)$ must be of the form

$$\psi(v_1 , v_2 , v_3) = -\sum_{i=1}^{3} \sum_{j=1}^{3} \beta_{ij} v_i v_j + \sum_{1}^{3} \alpha_i v_i + \Delta,$$

where α_i and Δ are constants. Inserting this result into $(1 - 1)$, one finds that

$$\ln \omega(\mathbf{v}) = -\sum_{j=1}^{3} \sum_{j=1}^{3} \beta_{ij}(1 - \gamma^2)v_i v_j + \sum_{1}^{3} \alpha_i(1 + \gamma)v_i + \Delta'. \tag{1-3}$$

But, since the distribution of small particles is assumed to be isotropic, one has

$$\omega(\mathbf{v}) = \omega(|\ \mathbf{v}\ |) = \omega((v_1^2 + v_2^2 + v_3^2)^{1/2}). \tag{1-4}$$

The only possible way $(1 - 3)$ can satisfy this condition is for

$$\beta_{ij} = \beta \delta_{ij} , \qquad \alpha_i = 0. \tag{1-5}$$

Hence

$$\mathcal{A}(v) = \mathcal{A}_0 \exp \{-\beta v^2\}, \qquad \omega(v) = \omega_0 \exp \{-\beta(1 - \gamma^2)v^2\},$$

where $\mathcal{A}_0$ and ω_0 are constants.

Appendix 2

Solution of the Boltzmann Equation

It is desired to solve Eq. (11) subject to the condition (2). Two methods will be given. One procedure is to introduce the Fourier transform of $w(\mathbf{v}, t)$, i.e.,

$$T_w(\mathbf{k}, t) = \int \exp \{i\mathbf{k} \cdot \mathbf{v}\} w(\mathbf{v}, t) \, d\mathbf{v}$$

with

$$T_w(\mathbf{k}, 0) = \int \exp \{i\mathbf{k} \cdot \mathbf{v}\} \ \delta(\mathbf{v} - \mathbf{v}_0) \, d\mathbf{v} = \exp \{i\mathbf{k} \cdot \mathbf{v}_0\}.$$

Taking the Fourier transform of Eq. (11) one obtains the following equation for $T_w(\mathbf{k}, t)$:

$$\frac{\partial}{\partial t} T_w(\mathbf{k}, t) = A(\mathbf{k}) T_w(\gamma \mathbf{k}, t) - \frac{1}{\tau} T_w(\mathbf{k}, t), \tag{2-1}$$

where

$$A(\mathbf{k}) = \int \exp\{i\mathbf{k}\cdot\mathbf{v}\}\mathfrak{A}(\mathbf{v})\,d\mathbf{v} = \mathfrak{A}_0 \int \exp\{i\mathbf{k}\cdot\mathbf{v} - \beta v^2\}\,d\mathbf{v}$$

$$= \mathfrak{A}_0\left(\frac{\pi}{\beta}\right)^{3/2} \exp\left\{-\frac{k^2}{4\beta}\right\} = \frac{1}{\tau}\exp\left\{-\frac{k^2}{4\beta}\right\}.$$

It is now convenient to introduce the Laplace transform

$$L_w(\mathbf{k}, s) = \int_0^\infty \exp\{-st\}T_w(\mathbf{k}, t)\,dt;$$

then

$$\int_0^\infty \exp\{-st\}\frac{\partial}{\partial t}T_w(\mathbf{k}, t)\,dt = -T_w(\mathbf{k}, 0) + sL_w(\mathbf{k}, s) = -\exp\{i\mathbf{k}\cdot\mathbf{v}_0\} + sL_w(\mathbf{k}, s).$$

Thus, taking the Laplace transform of (2-1), one obtains the equation for $L_w(\mathbf{k}, s)$,

$$-\exp\{i\mathbf{k}\cdot\mathbf{v}_0\} + sL_w(\mathbf{k}, s) = \frac{1}{\tau}\exp\left\{-\frac{k^2}{4\beta}\right\}L_w(\gamma\mathbf{k}, s) - \frac{1}{\tau}L_w(\mathbf{k}, s).$$

This may be rearranged to give

$$L_w(\mathbf{k}, s) = \frac{1}{s + \tau^{-1}}\exp\{i\mathbf{k}\cdot\mathbf{v}_0\} + \frac{1}{\tau}\frac{1}{s + \tau^{-1}}\exp\left\{-\frac{k^2}{4\beta}\right\}L_w(\gamma\mathbf{k}, s). \tag{2-2}$$

The finite difference equation (2-2) may be solved by the following procedure: Replace $\mathbf{k}$ by $\gamma\mathbf{k}$. This yields

$$L_w(\gamma\mathbf{k}, s) = +\frac{1}{s + \tau^{-1}}\exp\{i\gamma\mathbf{k}\cdot\mathbf{v}_0\} + \frac{1}{\tau}\frac{1}{s + \tau^{-1}}\exp\left\{-\frac{\gamma^2k^2}{4\beta}\right\}L_w(\gamma^2\mathbf{k}, s). \tag{2-3}$$

Equation (2-3) may be used to eliminate $L_w(\gamma\mathbf{k}, s)$ from (2-2), yielding

$$\begin{aligned} L_w(k, s) = \frac{1}{s + \gamma^{-1}}\exp\{i\mathbf{k}\cdot\mathbf{v}_0\} &+ \frac{1}{\tau}\frac{\exp\{i\gamma\mathbf{k}\cdot\mathbf{v}_0 - k^2/4\beta\}}{(s + \tau^{-1})^2} \\ &+ \frac{1}{\tau}\frac{\exp\{-\gamma^2k^2/4\beta\}}{s + \tau^{-1}}L_w(\gamma^2\mathbf{k}, s). \end{aligned} \tag{2-4}$$

Replacing $\mathbf{k}$ by $\gamma^2\mathbf{k}$ in (2-2), the resulting equation may be used to eliminate $L_w(\gamma^2\mathbf{k}, s)$ from (2-4). Continuing in this fashion yields the solution

$$L_w(\mathbf{k}, s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\tau^n}\frac{\exp\{i\gamma^n\mathbf{k}\cdot\mathbf{v}_0 - (k^2/4\beta)\Delta_n\}}{(s + \tau^{-1})^{n+1}},$$

where

$$\Delta_n = \frac{1 - \gamma^{2n}}{1 - \gamma^2}.$$

It is to be noted that the series is absolutely convergent.

$T_w(\mathbf{k}, t)$ may readily be obtained from $L_w(\mathbf{k}, s)$ by taking the inverse Laplace transform. Using a Bromwich contour it is apparent that

$$
\begin{aligned}
T_w(\mathbf{k}, t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} L_w(\mathbf{k}, s)\, ds \\
&= \frac{1}{2\pi i} \int_{Br} e^{st} L_w(\mathbf{k}, s)\, ds \\
&= \text{Residue at } s = -\frac{1}{\tau} \text{ of } L_w(\mathbf{k}, s) \\
&= \sum_0^\infty (-1)^n \frac{\exp\{i\gamma^n \mathbf{k}\cdot\mathbf{v}_0 - (k^2/4\beta)\Delta_n\}}{\tau^n} \qquad (2\text{-}5) \\
&\qquad \times \left[\text{Residue at } s = -\frac{1}{\tau} \text{ of } \frac{e^{st}}{(s + \tau^{-1})^{n+1}}\right] \\
&= \sum_0^\infty (-1)^n \frac{\exp\{i\gamma^n \mathbf{k}\cdot\mathbf{v}_0 - (k^2/4\beta)\Delta_n\}}{\tau^n} \frac{(-t)^n}{n!} \exp\left\{-\frac{t}{\tau}\right\} \\
&= \left[\sum_0^\infty \frac{1}{n!}\left(\frac{t}{\gamma}\right)^n \exp\{i\gamma^n \mathbf{k}\cdot\mathbf{v}_0 - (k^2/4\beta)\Delta_n\}\right] \exp\left\{-\frac{t}{\tau}\right\}.
\end{aligned}
$$

The inverse Fourier transform may now be performed and this yields

$$
\begin{aligned}
w(\mathbf{v}, t) &= \frac{1}{(2\pi)^3} \int \exp\{-i\mathbf{k}\cdot\mathbf{v}\} T_w(\mathbf{k}, t)\, d\mathbf{k} \\
&= \frac{\exp\{-t/\tau\}}{(2\pi)^3} \sum_0^\infty \frac{1}{n!}\left(\frac{t}{\tau}\right)^n \int \exp\left\{i(\gamma^n\mathbf{v}_0 - \mathbf{v})\cdot\mathbf{k} - \frac{k^2}{4\beta}\Delta_n\right\} d\mathbf{k} \\
&= \exp\left\{-\frac{t}{\tau}\right\}\left[\delta(v - v_0) + \sum_1^\infty \frac{1}{n!}\left(\frac{t}{\tau}\right)^n \left(\frac{\beta}{\pi\Delta_n}\right)^{3/2} \exp\left\{-\frac{\beta}{\Delta_n}(\mathbf{v} - \gamma^n\mathbf{v}_0)^2\right\}\right]. \qquad (2\text{-}6)
\end{aligned}
$$

This solution may be verified by substitution.

Equation (11) may also be solved in the following way [4]. Consider the sequence of equations,

$$
\begin{aligned}
\frac{\partial w_0(\mathbf{v}, t)}{\partial t} &= \frac{-w_0(\mathbf{v}, t)}{\tau} \\
\frac{\partial w_1(\mathbf{v}, t)}{\partial t} &= \frac{-w_1(\mathbf{v}, t)}{\tau} + \int w_0(\mathbf{v}', t)\,\mathfrak{a}(\mathbf{v} - \gamma\mathbf{v}')\, d\mathbf{v}' \qquad (2\text{-}7) \\
&\cdots\cdots\cdots\cdots \\
\frac{\partial w_n(\mathbf{v}, t)}{\partial t} &= \frac{-w_n(\mathbf{v}, t)}{\tau} + \int w_{n-1}(\mathbf{v}', t)\,\mathfrak{a}(\mathbf{v} - \gamma\mathbf{v}')\, d\mathbf{v}', \text{ etc.,}
\end{aligned}
$$

subject to the initial conditions

$$
\begin{cases} w_0(\mathbf{v}, 0) = \delta(\mathbf{v} - \mathbf{v}_0) \\ w_i(\mathbf{v}, 0) = 0, \qquad i \neq 0. \end{cases} \qquad (2\text{-}8)
$$

Plainly,

$$w(\mathbf{v}, t) = \sum_0^\infty w_n(\mathbf{v}, t) \tag{2-9}$$

satisfies Eq. (11) and the condition $w(\mathbf{v}, 0) = \delta(\mathbf{v} - \mathbf{v}_0)$.

Then

$$w_0(\mathbf{v}, t) = \delta(\mathbf{v} - \mathbf{v}_0) \exp\{-t/\tau\}$$

and

$$w_n(\mathbf{v}, t) = \exp\left\{-\frac{t}{\tau}\right\} \int_0^t \exp\left\{\frac{s}{\tau}\right\} \int w_{n-1}(\mathbf{v}'', s)\mathfrak{A}(\mathbf{v} - \gamma\mathbf{v}'')\, d\mathbf{v}''\, ds \tag{2-10}$$

satisfy the equations and one need only evaluate the sequence of functions, $w_n(v, t)$. It is seen from this last equation that if $w_{n-1}(\mathbf{v}, t)$ is a product of a function of $\mathbf{v}$ and a function of t, $w_n(\mathbf{v}, t)$ is also such a product. Since w_0 has such a form, all our w_n decompose in this way. Assume $w_n(\mathbf{v}, t) = U_n(\mathbf{v})g_n(t)$. Then

$$g_n(t) = \exp\left\{-\frac{t}{\tau}\right\} \int_0^t \exp\left\{\frac{s}{\tau}\right\} g_{n-1}(s)\, ds$$

and

$$U_n(\mathbf{v}) = \mathfrak{A}_0 \int U_{n-1}(\mathbf{v}') \exp\{-\beta(\mathbf{v} - \gamma\mathbf{v}')^2\}\, d\mathbf{v}'. \tag{2-11}$$

It is now easy to see that

$$g_n(t) = \frac{t^n}{n!} \exp\left\{-\frac{t}{\tau}\right\}. \tag{2-12}$$

$U_n(\mathbf{v})$ has the form $\alpha_n \exp\{-\beta_n(\mathbf{v} - \boldsymbol{\delta}_n)^2\}$, and α_n, β_n, $\boldsymbol{\delta}_n$ are connected by recursion relations derived from Eq. (2-11), stating

$$\alpha_{n+1} \exp\{-\beta_{n+1}(v - \boldsymbol{\delta}_{n+1})^2\} = \mathfrak{A}_0 \int \alpha_n \exp\{-\beta_n(\mathbf{v}' - \boldsymbol{\delta}_n)^2\} \exp\{-\beta(\mathbf{v} - \gamma\mathbf{v}')^2\}\, d\mathbf{v}'.$$

This gives

$$\alpha_{n+1} = \left[\frac{\pi}{\beta_n + \gamma^2\beta}\right]^{3/2} \mathfrak{A}_0\alpha_n \quad \text{with} \quad \alpha_1 = \mathfrak{A}_0$$

$$\beta_1 = B$$

$$\beta_{n+1} = \frac{\beta\beta_n}{\beta_n + \gamma^2\beta} \qquad \delta_1 = v_0 \tag{2-13}$$

$$\boldsymbol{\delta}_{n+1} = \gamma\boldsymbol{\delta}_n .$$

$$\therefore \boldsymbol{\delta}_n = \gamma^n\mathbf{v}_0$$

$$\beta_n = \frac{\beta}{\Delta_n} \quad \text{where} \quad \Delta_n = \frac{\gamma^{2n} - 1}{\gamma^2 - 1} \tag{2-14}$$

$$\alpha_n = \frac{1}{\tau^n}\left(\frac{\beta}{\pi\Delta_n}\right)^{3/2}.$$

When these are substituted into the series of Eq. (2-9), the solution is again obtained.

A CASE OF COMBINED RADIAL AND AXIAL HEAT FLOW IN COMPOSITE CYLINDERS*

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Introduction. Although several problems of heat flow in composite cylinders have been studied, all the cases considered treat the heat flow in the radial direction only [1, 2, 3]. The case of combined radial and axial heat flow in composite cylinders presents an interesting boundary value problem which has also considerable significance in the theory of vibrations and propagation of electromagnetic waves [4, 5, 6]. In this paper, we consider a case of combined radial and axial heat flow in the unsteady state in finite cylinders composed of two coaxial parts of different materials. The temperature distribution in the cylinder at any instant under the assumed boundary and initial conditions has been obtained by the use of the Laplace transformation. The procedure is illustrated by a numerical calculation in a particular case.

The Problem. Composite cylinder made of two different materials, the inner cylinder $0 \leq r \leq a$ and the outer cylinder $a \leq r \leq b$ having thermal conductivity and diffusivity coefficients ϵ_1 and k_1 and ϵ_2 and k_2 respectively.† Boundary conditions: The flat ends of the cylinder $x = 0$ and $x = l$ kept at zero temperature with the outer surface insulated and perfect thermal contact at $r = a$ between the two coaxial parts. Initially the cylinder is assumed to be heated to constant unit temperature. Required the temperature distribution in the cylinder for any time $t > 0$ (see Fig. 1).

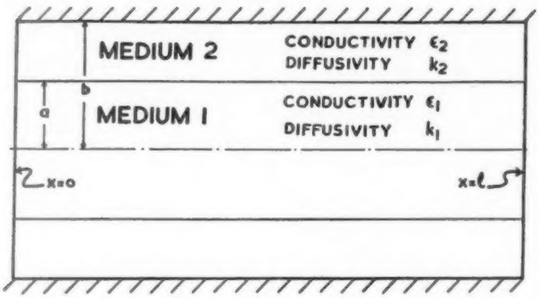

FIG. 1. Composite cylinder, the inner cylinder having conductivity and diffusivity coefficients ϵ_1 and k^1 respectively and the outer ϵ_2 and k_2.

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†Throughout this article the subscripts 1 and 2 refer to the regions $0 \leq r \leq a$ and $a \leq r \leq b$ respectively.

Method of solution. The boundary value problem for the temperatures u_1 and u_2 in the inner and outer cylinders may be stated as follows:

$$\frac{\partial u_1}{\partial t} = k_1\left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{\partial^2 u_1}{\partial x^2}\right), \qquad 0 \le r \le a, \tag{1}$$

$$\frac{\partial u_2}{\partial t} = k_2\left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} + \frac{\partial^2 u_2}{\partial x^2}\right), \qquad a \le r \le b, \tag{2}$$

with the initial conditions

$$u_1(r, x, 0) = u_2(r, x, 0) = 1, \tag{3}$$

and the boundary conditions

$$u_1(r, 0, t) = u_2(r, 0, t) = 0, \tag{4}$$

$$u_1(r, l, t) = u_2(r, l, t) = 0, \tag{5}$$

$$u_1(a, x, t) = u_2(a, x, t), \tag{6}$$

$$\epsilon_1\, \partial u_1(a, x, t)/\partial r = \epsilon_2\, \partial u_2(a, x, t)/\partial r, \tag{7}$$

$$\epsilon_2\, \partial u_2(b, x, t)/\partial r = 0. \tag{8}$$

We require the temperature functions $u_1(r, x, t)$ and $u_2(r, x, t)$ satisfying the equations from (1) to (8) inclusive.
Let

$$U_1(r, x, s) = \int_0^\infty u_1(r, x, t)e^{-st}\, dt$$

$$U_2(r, x, s) = \int_0^\infty u_2(r, x, t)e^{-st}\, dt$$

be the Laplace transforms of u_1 and u_2. The transforms $U_1(r, x, s)$ and $U_2(r, x, s)$ will then satisfy the equations

$$\frac{\partial^2 U_1}{\partial r^2} + \frac{1}{r}\frac{\partial U_1}{\partial r} + \frac{\partial^2 U_1}{\partial x^2} - \frac{s}{k_1} U_1 = -\frac{1}{k_1}, \tag{9}$$

$$\frac{\partial^2 U_2}{\partial r^2} + \frac{1}{r}\frac{\partial U_2}{\partial r} + \frac{\partial^2 U_2}{\partial x^2} - \frac{s}{k_2} U_2 = -\frac{1}{k_2}, \tag{10}$$

and the boundary conditions

$$U_1(r, 0, s) = U_2(r, 0, s) = 0, \tag{11}$$

$$U_1(r, l, s) = U_2(r, l, s) = 0, \tag{12}$$

$$U_1(a, x, s) = U_2(a, x, s), \tag{13}$$

$$\epsilon_1\, \partial U_1(a, x, s)/\partial r = \epsilon_2\, \partial U_2(a, x, s)/\partial r, \tag{14}$$

$$\partial U_2(b, x, s)/\partial r = 0. \tag{15}$$

In order that $U_1(r, x, s)$ and $U_2(r, x, s)$ may vanish at $x = 0$ and $x = l$ as required by the boundary conditions (11) and (12), we expand U_1 and U_2 as well as the constants $-1/k_1$ and $-1/k_2$ on the right hand side of equations (9) and (10) in Fourier sine series

$$U_1(r, x, s) = \sum_n V_{1n}(r, s) \sin (n\pi x/l), \qquad n = 1, 3, 5, \cdots,$$

$$U_2(r, x, s) = \sum_n V_{2n}(r, s) \sin (n\pi x/l), \qquad n = 1, 3, 5, \cdots,$$

$$-\frac{1}{k_1} = \sum_n b_{1n} \sin (n\pi x/l), \qquad n = 1, 3, 5, \cdots,$$

$$-\frac{1}{k_2} = \sum_n b_{2n} \sin (n\pi x/l), \qquad n = 1, 3, 5, \cdots.$$

For the sake of brevity, let

$$\alpha_n^2 = s/k_1 + n^2\pi^2/l^2 \tag{16}$$

$$\beta_n^2 = s/k_2 + n^2\pi^2/l^2 \tag{17}$$

The radial functions V_{1n} and V_{2n} now satisfy respectively the equations

$$\frac{\partial^2 V_{1n}}{\partial r^2} + \frac{1}{r}\frac{\partial V_{1n}}{\partial r} - \alpha_n^2\left(V_{1n} + \frac{b_{1n}}{\alpha_n^2}\right) = 0,$$

$$\frac{\partial^2 V_{2n}}{\partial r^2} + \frac{1}{r}\frac{\partial V_{2n}}{\partial r} - \beta_n^2\left(V_{2n} + \frac{b_{2n}}{\beta_n^2}\right) = 0.$$

These have the solutions

$$V_{1n}(r, s) = A_n I_0(\alpha_n r) - \frac{b_{1n}}{\alpha_n^2},$$

$$V_{2n}(r, s) = B_n I_0(\beta_n r) + C_n K_0(\beta_n r) - \frac{b_{2n}}{\beta_n^2},$$

where I_0 and K_0 are respectively the modified Bessel functions of the first and the second kinds of zeroth order. The constants A_n, B_n, etc. are now to be determined from the boundary conditions (13), (14) and (15). Thus we obtain

$$U_1(r, x, s) = \sum_n \left[\epsilon_2\beta_n\left(\frac{b_{1n}}{\alpha_n^2} - \frac{b_{2n}}{\beta_n^2}\right)\frac{\{I_1(\beta_n a)K_1(\beta_n b) - I_1(\beta_n b)K_1(\beta_n a)\}I_0(\alpha_n r)}{\Delta_n(s)} - \frac{b_{1n}}{\alpha_n^2}\right]\sin\frac{n\pi x}{l}, \tag{18}$$

$$U_2(r, x, s) = \sum_n \left[\epsilon_1\alpha_n\left(\frac{b_{1n}}{\alpha_n^2} - \frac{b_{2n}}{\beta_n^2}\right)\frac{\{I_0(\beta_n r)K_1(\beta_n b) - I_1(\beta_n b)K_0(\beta_n r)\}I_1(\alpha_n a)}{\Delta_n(s)} - \frac{b_{2n}}{\beta_n^2}\right]\sin\frac{n\pi x}{l}, \tag{19}$$

where

$$\begin{aligned}\Delta_n(s) = \epsilon_2\beta_n I_0(\alpha_n a)[I_1(\beta_n a)K_1(\beta_n b) - I_1(\beta_n b)K_1(\beta_n a)]\\ - \epsilon_1\alpha_n I_1(\alpha_n a)[I_0(\beta_n a)K_1(\beta_n b) + I_1(\beta_n b)K_0(\beta_n a)].\end{aligned} \tag{20}$$

The temperature distribution functions $u_1(r, x, t)$ and $u_2(r, x, t)$ may be now obtained from $U_1(r, x, s)$ and $U_2(r, x, s)$ by the inversion integrals [7]

$$u_1(r, x, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} U_1(r, x, s)e^{st}\,ds \tag{21}$$

$$u_2(r, x, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} U_2(r, x, s)e^{st}\,ds \tag{22}$$

The integrals in (21) and (22) may be expressed as $2\pi i$ times the sum of the residues of the corresponding integrands at their poles. In evaluating the residue of $U_1(r, x, s)$ exp (st) it will be noted that the first term in $U_1(r, x, s)$ has got the factor $\alpha_n^2\beta_n\Delta_n(s)$ in the denominator. It will be seen further that $\beta_n = 0$ does not give rise to a pole since the expression remains finite (on account of the singularity of K_1 at the origin) as $\beta_n \to 0$ so that the only poles are those due to $\alpha_n^2 = 0$ and $\Delta_n = 0$. However, the residue of the first term at $\alpha_n = 0$ cancels with that due to the second term and hence the only significant poles are those of $\Delta_n = 0$. Similar remarks apply in evaluating the residue of $U_2(r, x, s)$ exp (st). One obtains therefore

$$\begin{aligned}u_1(r, x, t) = \sum_n \frac{4\pi\epsilon_2}{l^2}\frac{k_1 - k_2}{k_1k_2} n \sin\frac{n\pi x}{l}\\ \cdot \sum_i \frac{\{I_1(\beta_{ni}a)K_1(\beta_{ni}b) - I_1(\beta_{ni}b)K_1(\beta_{ni}a)\}I_0(\alpha_{ni}r)}{\alpha_{ni}^2\beta_{ni}\Delta_n'(\lambda_{ni})} e^{\lambda_{ni}t},\end{aligned} \tag{23}$$

$$\begin{aligned}u_2(r, x, t) = \sum_n \frac{4\pi\epsilon_1}{l^2}\frac{k_1 - k_2}{k_1k_2} n \sin\frac{n\pi x}{l}\\ \cdot \sum_i \frac{\{I_0(\beta_{ni}r)K_1(\beta_{ni}b) - I_1(\beta_{ni}b)K_0(\beta_{ni}r)\}I_1(\alpha_{ni}a)}{\alpha_{ni}\beta_{ni}^2\Delta_n'(\lambda_{ni})} e^{\lambda_{ni}t},\end{aligned} \tag{24}$$

where λ_{ni} are the zeroes of $\Delta_n(s) = 0$, $(n = 1, 3, 5, \cdots)$ and α_{ni} and β_{ni} are the corresponding values defined by the relations (16) and (17) when s has the values λ_{ni}.

The zeroes of Δ_n may be obtained from equation (20) by solving the equation

$$\frac{\epsilon_1 a_n}{\epsilon_2\beta_n}\frac{I_1(\alpha_n a)}{I_0(\beta_n a)} = \frac{I_1(\beta_n a)K_1(\beta_n b) - I_1(\beta_n b)K_1(\beta_n a)}{I_0(\beta_n a)K_1(\beta_n b) + I_1(\beta_n b)K_0(\beta_n a)} \tag{25}$$

graphically. The equation (25) may be transformed into a form more suitable for numerical work by the substitutions

$$\alpha_n a = ix, \qquad \beta_n a = iy, \qquad \beta_n b = i\rho y, \qquad (\rho = b/a),$$

x and y being related on account of (16) and (17) by the equation

$$y^2 = \sigma^2(x^2 + n^2\pi^2a^2\delta^2/l^2), \tag{26}$$

where $\sigma^2 = k_1/k_2$ and $\delta^2 = (k_1 - k_2)/k_1$ are dimensionless constants. Introducing another dimensionless constant $\sigma'^2 = \epsilon_1/\epsilon_2$ and transforming the I and K functions into the corresponding J and Y functions* by the relations

$$I_\nu(iz) = i^\nu J_\nu(z),$$

$$K_\nu(iz) = i^{-\nu+1}[-J_\nu(z) + iY_\nu(z)]\pi/2,$$

we obtain in place of (25)

$$\sigma'^2 x \frac{J_1(x)}{J_0(x)} = y \frac{J_1(\rho y)Y_1(y) - J_1(y)Y_1(\rho y)}{J_1(\rho y)Y_0(y) - J_0(y)Y_1(\rho y)}. \tag{27}$$

Equation (27) has real roots and may be solved by plotting the right and left hand sides as functions of x. (Note that y on the right hand side is not the corresponding ordinate, but determined by (26)).

Let

$$x = \xi_{nj} \qquad j = 1, 2, 3, \cdots; \qquad n = 1, 3, 5, \cdots$$

be the roots of equation (27), the double subscript indicating that ξ_{nj} is the jth root of $\Delta_n = 0$. Let the corresponding values of $\alpha_{nj}a$, etc. be

$$\alpha_{nj}a = i\xi_{nj}\,, \qquad \beta_{nj}a = i\sigma\eta_{nj}\,, \qquad \beta_{nj}b = i\rho\sigma\eta_{nj}\,,$$

where

$$\eta_{nj}^2 = \xi_{nj}^2 + (n\pi a\delta/l)^2 \text{ by equation (26). Then}$$

$$\lambda_{nj} = -(\xi_{nj}^2/a^2 + n^2\pi^2/l^2)k_1\,.$$

$u_1(r, x, t)$ and $u_2(r, x, t)$ can be now expressed in terms of ξ_{nj} and η_{nj} as follows:

$$u_1(r, x, t) = \frac{8\pi a^3 \epsilon_2}{l^2}\frac{k_1 - k_2}{k_1\sigma}\sum_n n \sin\frac{n\pi x}{l}\sum_j \frac{F_{11}(\rho\sigma\eta_{nj}\,, \sigma\eta_{nj})J_0(\xi_{nj}r/a)}{\xi_{nj}^2\eta_{nj}D_n(\lambda_{nj})} e^{\lambda_{nj}t}, \tag{28}$$

$$u_2(r, x, t) = \frac{8\pi a^3 \epsilon_1}{l^2}\frac{k_1 - k_2}{k_1\sigma^2}\sum_n n \sin\frac{n\pi x}{l}\sum_j \frac{F_{10}(\rho\sigma\eta_{nj}\,, \sigma\eta_{nj}r/a)J_1(\xi_{nj})}{\xi_{nj}\eta_{nj}^2 D_n(\lambda_{nj})} e^{\lambda_{nj}t}, \tag{29}$$

where

$$\begin{aligned} D_n(\lambda_{nj}) = \frac{\epsilon_2 a}{\sigma}\left(\frac{\eta_{nj}}{\xi_{nj}} - \sigma'^2\frac{\xi_{nj}}{\eta_{nj}}\right)&J_1(\xi_{nj})F_{11}(\rho\sigma\eta_{nj}\,, \sigma\eta_{nj}) \\ &+ \frac{\epsilon_1 b}{\sigma}\frac{\xi_{nj}}{\eta_{nj}}J_1(\xi_{nj})F_{00}(\rho\sigma\eta_{nj}\,, \sigma\eta_{nj}) \\ &+ \epsilon_2 b J_0(\xi_{nj})F_{10}(\sigma\eta_{nj}\,, \rho\sigma\eta_{nj}) \\ &+ \frac{\epsilon_1 a}{\sigma'^2}\left(1 - \frac{\sigma'^2}{\sigma^2}\right)J_0(\xi_{nj})F_{01}(\sigma\eta_{nj}\,, \rho\sigma\eta_{nj}), \end{aligned} \tag{30}$$

$$F_{\mu\nu} = J_\mu(x)Y_\nu(y) - J_\nu(y)Y_\mu(x). \tag{31}$$

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*Ref. [3], Appendix III.

Verification of the solution.* As the series solution established by the Laplace transform method is purely formal, it is necessary to show that it satisfies all the conditions of the boundary value problem and is unique. It is obvious that the series solutions (28) and (29) for $u_1(r, x, t)$ and $u_2(r, x, t)$ respectively satisfy the boundary conditions (4) and (5). It is also seen that the boundary conditions (7) and (8) are satisfied by direct substitution of the expressions for u_1 and u_2, and (6) is satisfied on account of the relation $\Delta_n(\lambda_{nj}) = 0$. It only remains, therefore, to verify that the initial condition, viz., $u_1 = u_2 = 1$ for $t = 0$. This is done most conveniently with the contour integral form of the solutions (21) and (22). Consider u_1 for example.
For $t = 0$ we have from (18) and (21)

$$u_1(r, x, 0) = \sum_n \frac{4}{n\pi} \sin \frac{n\pi x}{l} + \sum_n \frac{4}{n\pi} \sin \frac{n\pi x}{l} \cdot \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} A_n I_0(\alpha_n r)\, ds \tag{32}$$

where

$$A_n = \frac{\epsilon_2(k_1 - k_2)}{k_1 k_2} \left(\frac{n\pi}{l}\right)^2 \frac{I_1(\beta_n a)K_1(\beta_n b) - I_1(\beta_n b)K_1(\beta_n a)}{\beta_n \alpha_n^2 \Delta_n(s)} . \tag{33}$$

since

$$\sum_n \frac{4}{n\pi} \sin \frac{n\pi x}{l} = 1 \qquad \text{we may write} \qquad u_1 = 1 + v_1$$

where

$$v_1 = \sum_n \frac{4}{n\pi} \sin \frac{n\pi x}{l} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} A_n I_0(\alpha_n r)\, ds. \tag{34}$$

It thus suffices to show that $v_1 \equiv 0$. The path of integration is a straight line parallel to the imaginary axis such that all the poles of the integrand lie to the left of this line. As the poles λ_{nj} are all negative we can choose the path with any $\gamma > 0$. We shall choose γ large and positive. Since $\alpha_n^2 = s/k_1 + n^2\pi^2/l^2$ and $\beta_n^2 = s/k_2 + n^2\pi^2/l^2$ it is clear that $|\alpha_n^2|$ and $|\beta_n^2|$ will be large both for large ξ and large η. Further, if $k_1 > k_2$ (say) we have on the path of integration $|\alpha_n| < |\beta_n|$.

Replacing now the modified Bessel functions in equation (33) for A_n by their asymptotic expansions for large argument and retaining only the dominant terms in the numerator and denominator we find that the integral in (34) becomes, apart from a constant factor

$$n^2\left(\frac{a}{r}\right)^{1/2} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{-\alpha_n(a-r)}}{\epsilon_2\beta_n + \epsilon_1\alpha_n} \frac{ds}{(\alpha_n\beta_n)^{3/2}} .$$

It may be shown that the absolute value of this expression is less than

$$2(\epsilon_1 + \epsilon_2)\left(\frac{a}{r}\right)^{1/2} n^2 \exp\left\{-\left(\frac{\gamma}{2k_1} + \frac{n^2\pi^2}{2l^2}\right)(a - r)\right\} \int_0^\infty \frac{d\eta}{(P^2 + Q^2\eta^2)^{1/2} \, |\alpha_n|^{3/2} \, |\beta_n|^{1/2}}$$

Thus the absolute values of the terms of the series in (34) are majorized by

$$c_1\left(\frac{a}{r}\right)^{1/2} \exp\left\{-\frac{r}{2k_1}(a - r)\right\} \sum_n K(n) \exp\left\{-\frac{n^2\pi^2}{2l^2}(a - r)\right\}, \tag{35}$$

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*Ref. [3], Appendix I.

where $K(n)$ is $0(nu)$ with a fixed finite μ. The expression (35) shows that v_1 can be made arbitrarily small by making γ large. Hence it follows that $v_1 \equiv 0$. Similar reasoning shows that u_2 also satisfies the initial condition.

The proof of the uniqueness of the solution is well known and need not be repeated here.

A numerical example. To illustrate the numerical procedure, the temperature distribution in a composite cylinder having the following parameters is calculated

$$k_1 = 1, \qquad k_2 = 0.1, \qquad \epsilon_1 = 0.4, \qquad \epsilon_2 = 0.04$$

$$a = 1, \qquad b = 1.5, \qquad l = 10; \qquad u_1(r, x, 0) = u_2(r, x, 0) = 1$$

Equation (27) leading to the roots ξ_{nj} in this particular case is plotted in Fig. 2.

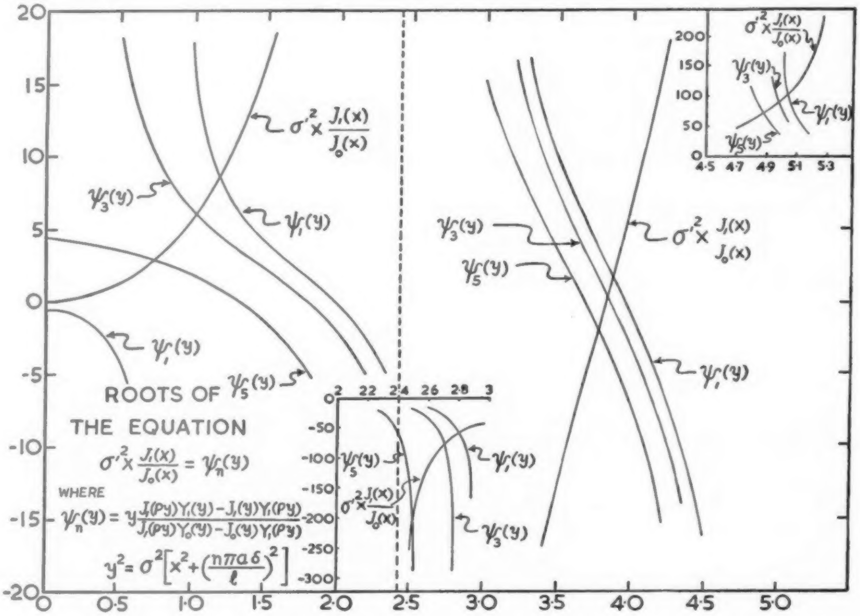

FIG. 2. Graphical solution of the first few roots of the eq. 27 for $n = 1, 3, 5$.

The first few roots are given in the table below.

n \ j	1	2	3	4
1	1.18	2.83	3.88	5.04
3	1.03	2.74	3.85	4.98
5	0.75	2.52	3.78	4.88

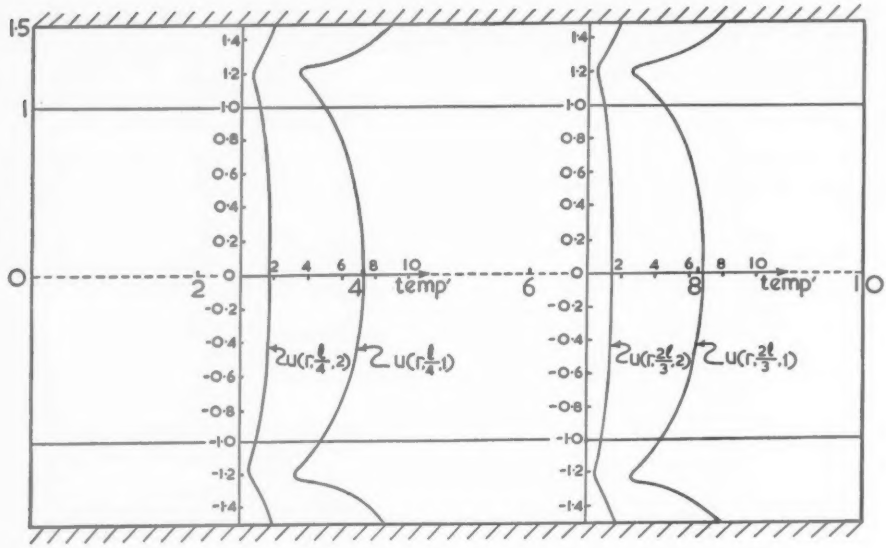

FIG. 3. Distribution of temperature in the composite cylinder along the radius at $x = l/4$ and $x = 2l/3$ for $t = 1$ and $t = 2$ secs.

The distribution of temperature along the radius at $x = 2l/3$ and $l/4$ for $t = 1$ sec. and 2 secs. is plotted in Fig. 3.

BIBLIOGRAPHY

1. J. C. Jaeger, *Heat conduction in composite circular cylinders*, Phil. Mag. (213) **32**, 324-335 (1941).
2. L. P. Smith, *Heat flow in an infinite solid bounded internally by a cylinder*, Jour. App. Phys. (6) **8**, 441-448 (1937).
3. H. S. Carslaw and J. C. Jaeger, *Conduction of heat in solids*, Oxford, 1947, Art. 119.
4. W. J. Jacobi, *Propagation of sound waves along liquid cylinders*, Jour. Acous. Soc. Amer. (2) 21, 120-127 (1949).
5. L. Pincherle, *Electromagnetic waves in metal tubes filled longitudinally with two dielectrics*, Phys. Rev. (5,6) **66**, 118-130 (1944).
6. R. D. Teasdale and T. J. Higgins, *Electromagnetic waves in circular wave guides containing two coaxial media*, Proc. National Electronics Conference (USA), **5**, 427-441 (1949).
7. R. V. Churchill, *Modern operational methods in engineering*, McGraw Hill, 1944, p. 157.

—NOTES—

ON THE RATE OF CONVERGENCE OF RELAXATION METHODS*

BY R. PLUNKETT (*The Rice Institute*)**

A recent paper by Frankel [1] gives the rates of convergence and a time estimate for the solution of finite approximations to Poisson's equations and the biharmonic equation by some of the standard iteration methods. This shows that in general the time required is prohibitive for a reasonably large number of points. It is well known that relaxation methods [2] are faster for hand computing but an estimate of rate of convergence is of interest for machine programming for the prospective use of digital calculators. In general the finite approximation to partial differential equations may be written:

$$Az + f = 0 \tag{1}$$

where z is an n-element column matrix of the unknown values, and n is the number of points taken in the region. A is an $n \times n$ square matrix, with a high degree of regularity; the elements of the main diagonal are the largest and most of the other elements are zero; f is a known column matrix. If z_m is the mth approximation to z, the matrix of the residuals is defined by

$$Az_m + f = R_m \tag{2}$$

Then one element of z_m is adjusted in such a way as to make the greatest reduction in the value of R_m. Thus, if e_i is a unit vector of A-space, i.e., e_i has zero for all its elements except the i-th element which is 1,

$$z_{m+1} = z_m - c_{m+1}e_i \tag{3}$$

So far the value of R_m has not been defined, but it is clear that if all the elements of R_m are reduced to zero $z_m = z$. The usual approach in the numerical solution by relaxation methods is intuitive, but it can easily be seen that if

$$c_{m+1} = \frac{e_i'R_m}{e_i'Ae_i} \tag{4}$$

then

$$e_i'R_{m+1} = 0.$$

Thus, this has the effect of reducing one element of R_{m+1} to zero even though it may actually increase the others. It has been shown [3], if $A = A'$ and is a positive definite form, that

$$H_m = \tfrac{1}{2}z_m'Az_m + z_n'f \tag{5}$$

will be reduced by such a step and the process must ultimately converge. It is difficult to get an estimate for H_m, however, and so this is not the easiest criterion to use. In our case we shall use the more customary standard of

$$| R_m | = (R_m'R_m)^{1/2}. \tag{6}$$

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From Eqs. (3) and (2)

$$R_{m+1} = R_m - c_{m+1}Ae_i \tag{7}$$

and

$$|R_{m+1}|^2 = |R_m|^2 - 2c_{m+1}R_m'Ae_i + c_{m+1}^2 e_i'A'Ae_i \tag{8}$$

with no restrictions on A. Thus

$$(|R_m| - |R_{m+1}|)(|R_m| + |R_{m+1}|) = 2c_{m+1}R_m'Ae_i - c_{m+1}^2 e_i'A'Ae_i\ . \tag{9}$$

Since in general $|R_m|$ and $|R_{m+1}|$ differ very little, letting $|R_m| - |R_{m+1}| = \Delta R_{m+1}$

$$\Delta R_{m+1} \approx \frac{2c_{m+1}R_m'Ae_i - c_{m+1}^2 e_i'A'Ae_i}{2\,|R_m|}. \tag{10}$$

Let the elements of R_m be random variables with a mean value of zero, a maximum value of ρ_m and a standard deviation of $b\rho_m$ where b is a number less than one. Then the mathematical expectation of $|R_m|$ is:

$$|R_m| = n^{1/2}b\rho_m\ . \tag{11}$$

The value of c_{m+1} is found from Eq. (4) by taking i such as to make c_{m+1} take its largest value. From Eq. (10) the mathematical expectation of $\Delta R_{m+1}/|R_m|$ can be found.

From Eq. (4),

$$c_{m+1} = \frac{e_i'R_m}{e_i'Ae_i} \approx \frac{\rho_m}{a_{ii}}$$

and

$$R_m'Ae_i \approx a_{ii}\rho_m\ .$$

where a_{ii} is a typical element of the main diagonal.

Likewise

$$e_i'A'Ae_i = la_{ii}^2\ ,$$

where l is a small number greater than one [for Poisson's equation $l = 1.25$, for the biharmonic $l = 1.72$].

Thus,

$$2c_{m+1}R_m'Ae_i \approx 2\rho_m^2\ ,$$

$$c_{m+1}^2 e_i'A'Ae_i \approx l\rho_m^2$$

or, the mathematical expectation of

$$\frac{\Delta R_{m+1}}{|R_m|} = \frac{\rho_m^2(2-l)}{2nb^2\rho_m^2} = \frac{1 - l/2}{b^2 n} = k\,\frac{1}{n}\,, \tag{12}$$

where k is a number not much greater than one unless b, the standard deviation ratio, is very small. Thus for any reasonable distribution of the elements of R_m, the probability of $\Delta R_{m+1}/|R_m|$ being very much greater than k/n is small, or the standard deviation of

this quantity must be of the same order of magnitude as the quantity itself. Since because of the way it is found the correlation among these ratios for different m is small, then

$$\frac{|R_{m+n}|}{|R_m|} = \prod_{i=1}^{n}\left(1 - \frac{\Delta R_{m+i+1}}{|R_{m+i}|}\right) \tag{13}$$

and

$$\log\frac{|R_{m+n}|}{|R_m|} = \sum_{i=1}^{n}\log\left(1 - \frac{\Delta R_{m+i+1}}{|R_{m+i}|}\right) = \sum_{i=1}^{n} - \frac{k}{n} - \frac{1}{2}\left(\frac{k}{n}\right)^2 - \cdots$$

Thus the mathematical expectation of

$$\log\frac{|R_{m+n}|}{|R_m|} = -k + o\left(\frac{1}{n}\right)$$

and a good working value for

$$r_n = \frac{|R_{m+n}|}{|R_m|} \text{ is } e^{-k} \tag{14}$$

for large n.

The ratio of the standard deviation of r_n to r_n is of the order of magnitude of $1/n^{\frac{1}{2}}$ times the ratio of the standard deviation of $1 - \Delta R_{m+1}/|R_m|$.
Since

$$1 - \frac{\Delta R_{m+1}}{|R_m|} \approx 1,$$

and the standard deviation has already been seen to be of the order of k/n, this ratio will also be of the order of k/n. Thus the standard deviation of r_n, which is even smaller by a factor of $1/n^{\frac{1}{2}}$, would introduce little chance that r_n can be greatly in error.

Since k is a number close to one, something like 20 such steps as are indicated in equation (14) will reduce the error as measured by $|R_m|$ to at most 10^{-6} of its original value. This is in general agreement with the usual experience in relaxation techniques [2]. Each such series of n steps requires about as much computation as one iteration of the whole matrix if the process of selection of c_{m+1} is ignored; this is of the order of dn arithmetical manipulations where d is a small whole number of the order of 10, depending on the number of terms of one row of A which differ from zero. If the operation of "compare" which is necessary to find a maximum value for c_m, takes e arithmetical operations, a total of about $20\ (e + d)n$ operations will be necessary to reduce $|R_m|$ to 10^{-6} of its original value. The presently contemplated methods for "compare" [4] would make e equal to n which makes the total about $20\ n^2$ for large n regardless of the nature of A. This may be compared with the best results obtained by Frankel for iteration methods of about $20\ n^{3/2}$ for Poisson's equation and about $20\ n^2$ for the biharmonic equation. Thus, there is no saving in a relaxation method, which is actually more difficult to program, unless the problem is more complicated than the biharmonic equation. However, there is a possibility of making the "compare" operation more rapid by proper design, which would reduce the above estimate. From a purely heuristic viewpoint it is clear that by any method, at least $h\ n$ operations are necessary to evaluate n points, where h is a small number greater than one; the difference between this and the above estimate is completely taken up by the "compare" operation.

REFERENCES

1. S. P. Frankel, *Convergence rates of iterative treatments of partial differential equations*, Math. Tables and Aids to Comp. **4**, 65 (1950).
2. R. V. Southwell, *Relaxation methods in physical sciences*, Oxford Press, 1948.
3. J. L. Synge, *A geometrical interpretation of the relaxation method*, Q. Appl. Math. **2**, 87 (1944).
4. H. D. Huskey, *Characteristics of the INA computer*, Math. Tables and Aids to Comp. **4**, 103 (1950).

THE STABILITY EQUATION WITH PERIODIC COEFFICIENTS*

By HIRSH COHEN (*Haifa, Israel*)

In a large number of physical problems involving periodic motion, dynamic stability considerations result in stability differential equations which have periodic coefficients. In particular, if the physical system is described by a non-linear second order ordinary differential equation, a second order equation of the Floquet type appears. That this is not an isolated case becomes apparent if one reviews the large volume of non-linear mechanics literature of the past few years. The problem to be discussed in this note is even more specialized than the one just introduced but the same review through the literature will reveal that it is an important case. This is the stability problem which results when the non-linear element has small effect on the system, i.e., when the resultant motion is near to the motion of the linearized system.

As an example consider the van der Pol equation

$$y'' - \epsilon(1 - y^2)y' + y = 0 \tag{1}$$

where primes refer to differentiation with respect to t. If y is taken to be of 0 (1) then $\epsilon \ll 1$. The usual stability considerations involve the addition of a small (of order ϵ) time-dependent function, $v(t)$, to an exact or approximate solution $y_0(t)$. On substitution into (1) of $y = y_0 + v(t)$, the equation of first order in $v(t)$ is

$$v'' - \epsilon(1 - y_0^2)v' + (1 + 2\epsilon y_0 y_0')v = 0. \tag{2}$$

If the solution is to be a periodic approximation to y, then y_0 is periodic and (2) represents an example of the general equation dealt with herein, namely

$$u'' + \epsilon p(t)u' + \epsilon\left(q_1(t) + \frac{1}{\epsilon}\right)u = 0, \tag{3}$$

where u is the disturbance function being used to "test" some system, and $p(t)$ and $q_1(t)$ are periodic functions of period $2\pi/\omega$.

It can be seen immediately that the Mathieu equation is a special case of (3). Furthermore, it would appear useful to remove the first order term in (3) and thus reduce it to at worst a Hill equation. This may be done by the substitution

$$u = v(t) \exp\left[-\frac{\epsilon}{2}\int p(t)\,dt\right]. \tag{4}$$

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with the resulting equation in v

$$v'' + v + \epsilon q_1(t)v = 0, \tag{5}$$

and is in fact the method employed by McLachlan [1].*

The idea here, however, is to work directly with (3). It was found in dealing with the stability of subharmonics of the forced van der Pol equation [2] that the transformation (4) and the resulting Hill equation (5) were cumbersome to work with when the desired information was only whether $u(t)$ was a function which increased, decreased, or remained periodic with time. (This, of course, is what is meant here by stability. If the small disturbance $u(t)$ grows with time, the original physical system is said to be unstable).

The following approach will be adopted: An analysis due originally to Poincare but used in the form given by Friedrichs [3] will be applied to (3) to discover if *periodic* analytic solutions to (3) exist. From the general Floquet theory [4] and the theorems of Haupt [5] for the Hill equation we are led to expect that these periodic solutions will form the boundaries between the stable and unstable regions. Once assured that there are *periodic* analytic solutions (analyticity in ϵ is implied by the general existence theorems; it is only the periodicity of these analytic solutions that is tested), the solution $u(t)$ is expanded in a power series in ϵ. This last named step will again produce a purely periodic solution but will also produce the conditions on $p(t)$, $q_1(t)$ and ω for which stable solutions exist. The feature of this analysis which may be novel is that it is shown that the periodic coefficients need not have exactly the linearized period of (3) but may be somewhat different [according to the form of $p(t)$ and $q_1(t)$] and still produce a stable solution.

It should be emphasized here that this is intended to provide a quantitative study of the special equation (3) dealt with, and even that only in the restricted region of ϵ small. Qualitative investigations begin with Liapounoff [6] and have been taken up by other authors.

In order to use Friedrich's approach directly, let $\varphi = \omega t$ and consider a phase shift δ in φ such that the equation (3) now written in the form

$$u'' + u = -\epsilon[p(\theta + \delta)u' + q_1(\theta + \delta)u]$$

has the special initial conditions

$$u(0) = A, \qquad u'(0) = 0.$$

Following Friedrichs we introduce the variable η such that the period of the resultant motion, T, equals $2\pi + \epsilon\eta(\epsilon)$. Friedrichs then seeks values of A, δ, and η, taken to be analytic in ϵ, such that periodicity conditions on the solutions near $\epsilon = 0$ are satisfied. The method is given fully in the textbook of Stoker [3, p. 233] and need not be repeated here. In the case of a linear equation, A is independent of ωt and is not involved in the periodicity considerations. The condition for periodicity of solutions finally obtained is:

$$\eta_0 = -\pi\left\{c_0 \mp \left[\frac{1}{4}(a_2 - d_2)^2 + \frac{1}{4}(c_2 + b_2)^2 - a_0^2\right]^{1/2}\right\}.$$

*Numbers in brackets refer to references given at the end of the paper.

Here η_0 is the first term in an expansion of $\eta(\epsilon)$ in powers of ϵ, and the c_0, a_0, a_2, b_2, c_2, d_2 are the coefficients in the Fourier series expansion of p and q_1, namely

$$p(t) = a_0 + \sum_{n=1}^{\infty} (a_n \sin n\omega t + b_n \cos n\omega t),$$

$$q_1(t) = c_0 + \sum_{n=1}^{\infty} (c_n \sin n\omega t + d_n \cos n\omega t).$$

Translating back into terms of ω, we have

$$\omega = 1 - \frac{\epsilon\eta_0}{2\pi} = 1 - \frac{\epsilon}{2}\left\{c_0 \mp \left[\frac{1}{4}(a_2 - d_2)^2 + \frac{1}{4}(c_2 + b_2)^2 - a_0^2\right]^{1/2}\right\}.$$

Thus periodic solutions will exist when the period of the coefficients is given by

$$T = \frac{2\pi}{\omega} = 2\pi\left(1 + \frac{\epsilon}{2}\left\{c_0 \mp \left[\frac{1}{4}(a_2 - d_2)^2 + \frac{1}{4}(c_2 + b_2)^2 - a_0^2\right]^{1/2}\right\}\right).$$

Let us now investigate (3) by the customary formal expansion in a power series in ϵ. With the change of variable $\tau = \omega t$, we have

$$u''(\tau) + \frac{\epsilon}{\omega} p(\tau)u'(\tau) + \frac{\epsilon}{\omega^2}\left(q_1(\tau) + \frac{1}{\epsilon}\right)u(\tau) = 0. \tag{6}$$

It is well known [4] that the fundamental solution to the Floquet equation is given by

$$u = e^{\lambda\tau} g(\tau),$$

where λ is a complex number and $g(\tau)$ has period 2π. Let us take λ, g, and ω to be analytic functions of the small parameter ϵ so that we may write the expansions

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots,$$

$$g = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \cdots,$$

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots.$$

Substituting these into (6), equations of any desired degree in ϵ are obtained. For the degree zero:

$$g_0'' + 2\lambda_0 g_0' + (\lambda_0^2 + 1) = 0.$$

Now if the condition that g be periodic is imposed, the g_n must be periodic; in particular, g_0 must have the linearized period, 2π. This will obtain if

$$\lambda_0 = \pm i, \qquad g_0 = c + e^{-2\lambda_0\tau}.$$

Here one of the integration constants has been taken to be unity with no loss in generality. Proceeding, the equation of first degree in ϵ can now be found making use of the zeroth order solutions, λ_0 and g_0. The result is

$$g_1'' + 2\lambda_0 g_1' = -(c + e^{-2\lambda_0\tau})[2\lambda_1\lambda_0 - 2\omega_1 + q_1 + \lambda_0 p] + 2\lambda_0 e^{-\lambda_0 2\tau}[2\lambda_1 + p]. \tag{7}$$

Since the solution g_1 is to be periodic, the terms on the right hand side of (7) which will produce solutions that are non-periodic must be eliminated. This represents the imposition of the periodicity condition as a boundary condition. The method, often termed the casting out of secular terms, was given by Poincare and used extensively by

Duffing and many others. The terms on the right hand side of (7) which give rise to secular solutions are the constant terms and the terms with $e^{-2\lambda_0\tau}$ as multiplier. Again we use an expansion of the periodic functions:

$$p(\tau) = a_0 + \sum_{n=1}^{\infty} a_n \sin n\tau + b_n \cos n\tau,$$

$$q_1(\tau) = c_0 + \sum_{n=1}^{\infty} c_n \sin n\tau + d_n \cos n\tau.$$

Collecting constants and coefficients of $e^{-2\lambda_0\tau}$, we have

$$-c[2\lambda_1\lambda_0 - 2\omega_1 + c_0 + \lambda_0 a_0] - \frac{c_2}{2i} + \frac{\lambda_0 a_0}{2i} - \frac{d_2}{2} + \frac{\lambda_0 b_2}{2} = 0,$$

$$-[-2\lambda_0\lambda_1 - \lambda_0 a_0 - 2\omega_1 + c_0] - c\left[-\frac{c_2}{2i} + \frac{d_2}{2} - \frac{\lambda_0 a_2}{2i} + \frac{b_2\lambda_0}{2}\right] = 0.$$

Eliminating c we obtain

$$2\lambda_1 = -a_0 \pm \left[\frac{1}{4}(c_2 + b_2)^2 + \frac{1}{4}(a_2 - d_2)^2 - (c_0 - 2\omega_1)^2\right]^{1/2}. \tag{8}$$

To check the previous result, we set $\lambda_1 = 0$ and obtain

$$\omega_1 = \frac{c_0}{2} \mp \frac{1}{2}\left[\frac{1}{4}(c_2 + b_2)^2 + \frac{1}{4}(a_2 - d_2)^2 - a_0^2\right]^{1/2}.$$

Since $\omega = 1 + \epsilon\omega_1$ in this approximation and $\omega = 1 - \epsilon\eta_0/2\pi$ has been used, then $\omega_1 = -\eta_0/2\pi$. The result is

$$\eta_0 = -\pi\left(c_0 \pm \left[\frac{1}{4}(c_2 + b_2)^2 + \frac{1}{4}(a_2 - d_2)^2 - a_0^2\right]^{1/2}\right).$$

Returning to (8), it is observed:

1) if

$$(c_0 - 2\omega_1)^2 \geq \frac{1}{4}[(c_2 + b_2)^2 + (a_2 - d_2)^2],$$

stability is determined entirely by a_0

a) $a_0 < 0$, $\lambda_1 > 0$, and the resulting $u(t)$ increases with time

b) $a_0 > 0$, $\lambda_1 < 0$, and $u(t)$ decreases with time.

2) if

$$(c_0 - 2\omega_1)^2 < \frac{1}{4}\{(c_2 + b_2)^2 + (a_2 - d_2)^2\},$$

stability is determined by

$$-\frac{a_0}{2} + \frac{1}{2}\left[\frac{1}{4}\{(c_2 + b_2)^2 + (a_2 - d_2)^2\} - (c_0 - 2\omega_1)^2\right]^{1/2}.$$

Let us consider as an example to explicate the above work the equation (2) where

y_0 will be taken to be A sin t. (This is exactly the problem solved by McLachlan in [1], p. 190). Then

$$p(t) = \left[\left\{\frac{A^2}{2} - 1\right\} - \frac{A^2}{2}\cos 2t\right],$$

$$q_1(t) = A^2 \sin 2t,$$

$$a_0 = \frac{A^2}{2} - 1.$$

$$a_2 = c_0 = d_2 = 0, \qquad c_2 = A^2; \qquad b_2 = -\frac{A^2}{2}$$

Notice that for periodicity we would need

$$\omega_1 = \frac{1}{2}\left(\frac{A^4}{16} - \frac{A^4}{4} + A^2 - 1\right)^{1/2} = 0,$$

since here $\omega = 1$. If $A^2 = 4$, then this condition is satisfied. Further, using (8) since $c_0 - 2\omega_1 = 0$ and $\frac{1}{4}\{(c_2 + b_2)^2 + (a_2 - d_2)^2\} > 0$

$$\lambda_1 = -\frac{A^2}{8} + \frac{1}{2}.$$

Thus $\lambda_1 < 0$ for $A^2 > 4$ and >0 for $A^2 < 4$.

This example as was remarked is given by McLachlan [1] and was presented merely to show the ease in which stability characteristics may be obtained once λ_1 is computed in terms of a_0, a_2, b_2, c_2, and d_2.

References

1. N. McLachlan, *Ordinary non-linear differential equations*, Oxford, 1950.
2. H. G. Cohen, *Subharmonic synchronization of the forced Van der Pol equation* (To appear in the Proceedings of the Colloquium on non-linear vibrations, Île de Porquerolles, August, 1951.
3. J. J. Stoker, *Non-linear vibrations*, Interscience, New York, 1950. (see especially Appendix I).
4. E. L. Ince, *Ordinary differential equations*, Dover, New York.
5. M. J. O. Strutt, *Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik*, Julius Springer, Berlin, 1932.
6. A. Liapounoff, *Problème general de la stabilité du mouvement*, Princeton University Press, Princeton, 1949.

ON THE RELATIONSHIP BETWEEN THE MARTIENSSON AND DUFFING METHODS FOR NONLINEAR VIBRATIONS*

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The background for a number of one-term approximation methods and their application to forced nonlinear vibrations has recently been discussed by Schwesinger.[1]

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[1]G. Schwesinger, *On one-term approximations of forced nonharmonic vibrations*, J. Appl. Mech. **17**, 202-208 (1950). Note that he attributes to Rüdenberg the method that is designated here as Martiensson's method.

As he points out, such one-term approximations may be useful in problems of analysis when the response is close to sinusoidal, as is frequently the case for small nonlinearities. They are not the ultimate tool, however, for one can always go to approximations which contain the higher harmonic terms as well. The situation is different for problems of nonlinear synthesis. There, the nature of the problem is such that approximations of more than one term are virtually prohibited, and one is forced to accept answers that are given by the one-term methods. Thus, there is a body of problems for which such methods have intrinsic importance.

The requirements of synthesis have led to a re-examination of some of these one-term methods. For this purpose, it is important to know in a general way whether the use of the various methods leads to different synthesis. In particular, one should know which of the methods are really independent of one another, in order to know how many synthesis possibilities must be examined. It was found that the results obtained by the methods known by the names of Martiensson (or Rüdenberg) and Duffing[2] are not independent. For the class of equations to which both methods are applicable, there is a simple relationship between their results. The purpose of this note is to develop this relationship.

The fact that the results of the two methods are simply related is the more striking because their rationales are so different. Briefly, these are as follows. In the Martiensson method, since an assumed one-term solution does not generally satisfy the differential equation identically, one forces satisfaction at two points, say $t = 0$ and at the quarter-period. In this way, an algebraic equation is obtained for the amplitude of the one-term solution. To use the Duffing method, one writes the equation in a form for iteration, and puts an assumed one-term solution into this equation as a first approximation. In order to enforce the requirement that the next approximation be periodic, one is obliged to choose a certain coefficient equal to zero. This again leads to an algebraic equation for the amplitude. As a development of the Lindstedt perturbation method, the Duffing method is perhaps the more rational procedure of the two. (It is fundamentally different, of course, in that it permits extension to higher approximations.)

Now let us consider the methods in greater detail. They are usually illustrated only for systems with one degree of freedom, but it is not difficult to extend them to certain higher order systems. The Martiensson method, as ordinarily applied, is limited to systems without dissipation. The most general equation to which it seems appropriate, at least without major changes in formalism, is

$$\Omega^2 \ddot{x} + Lx + \nu f(x) = F \sin t. \tag{1}$$

Here, $x(t)$ and f are $(n \times 1)$ column matrices, F is a constant $(n \times 1)$ matrix, and L is a constant $(n \times n)$ matrix. Both Ω^2 and ν are scalar parameters, the former being related to the frequency of excitation and the latter being the nonlinearity parameter. Suppose that $f(x)$ is a continuous odd function, with $f(0) = 0$. We will seek a periodic solution with period 2π.

Let us apply the Martiensson method formally to Eq. (1). We assume an approximation solution $x^{(0)} = a \sin t$ (a being a column matrix of amplitudes), which satisfies

[2] We restrict our consideration to the first approximation by this method, in effect making it a one-term method.

the equation at time $t = 0$. If we require that it also satisfy the equation at $t = \pi/2$, we obtain

$$(\Omega^2 - L)a - \nu f(a) + F = 0. \tag{2}$$

On the other hand, let us write Eq. (1) in a form for iteration as

$$\Omega^2(\ddot{x}^{(1)} + x^{(1)}) = (\Omega^2 - L)x^{(0)} - \nu f(x^{(0)}) + F \sin t.$$

Using the same approximation as before,

$$\Omega^2(\ddot{x}^{(1)} + x^{(1)}) = [(\Omega^2 - L)a + F] \sin t - \nu f(a \sin t). \tag{3}$$

Since f is a periodic function of t with period 2π, we can write

$$f(a \sin t) = \mathfrak{F}_1(a) \sin t + \sum_{m=2}^{\infty} \mathfrak{F}_{2m-1} \sin (2m - 1)t.$$

Following the Duffing iteration method, we require that no secular term arise when Eq. (3) is solved for $x^{(1)}$. This means that we must put the coefficient of sin t in Eq. (3) equal to zero, namely

$$(\Omega^2 - L)a - \nu\mathfrak{F}_1(a) + F = 0. \tag{4}$$

Equation (4) is the analog of Eq. (2), and is identical except that the first Fourier coefficient of $f(a \sin t)$ replaces the value $f(a)$.

This result has limited usefulness in its general form. However, in systems which contain only one nonlinear element, f is particularly simple. It has the representation $f = M\varphi(x_i)$, where M is a column matrix and φ is a scalar function of only one of the x-components. As perhaps the simplest possible example, we may consider the case where $\varphi(x_i) = x_i^m$, i.e. a simple power function. Equations (2) and (4) become respectively

$$(\Omega^2 - L)a - \nu a_i^m + F = 0$$

$$(\Omega^2 - L)a - c\nu a_i^m + F = 0$$

where c is the constant whose value is the first Fourier sine coefficient of $\sin^m t$. For example, if $m = 3$, then $c = 3/4$.

The result of this simple special case has a very useful implication in synthesis problems. It means that if a synthesis is attempted for a system containing a single nonlinear element which obeys a power law, the system being described by Eq. (1), then the procedure follows identical paths for the Martiensson and Duffing methods. It is known at once that the optimum nonlinearity by one of the methods, say the Duffing, is just $1/c$ times as large as those predicted by the other, and that optimum values of any parameters contained in L are identical by the two methods. Nothing should be inferred, of course, as to which method is better as a one-term approximation, since this is an entirely separate problem.

NOTE ON AERODYNAMIC HEATING WITH A VARIABLE SURFACE TEMPERATURE*

By A. E. BRYSON (*Hughes Aircraft Company*)

Emmons [1], has considered the problem of an insulated flat plate of infinite extent started impulsively from rest in a viscous, incompressible fluid. One of the interesting results of his analysis was a simple expression for the temperature recovery factor at the plate surface. Another interesting result can be obtained from the same problem by considering, instead of an insulated plate, a plate with a surface temperature that is a given function of time.

As Emmons has shown, letting μ, the viscosity coefficient of the fluid, ρ, the fluid density, and k, the thermal conductivity of the fluid, be constant, it follows from the equations of motion and the boundary conditions that the pressure is constant and the velocity normal to the plate is zero. The momentum equations reduce to:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{1}$$

where u is the velocity component of the fluid parallel to the surface of the plate, t is the time, y is the distance perpendicular to the plate surface, and $\nu = \mu/\rho$. The energy equation becomes:

$$\rho C_P \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial y^2} = \mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{2}$$

where T is the fluid temperature and Cp is the fluid specific heat per unit mass at constant pressure. These equations are to be solved with the following boundary and initial conditions:

$$u(0, t) = 0, \qquad u(y, 0) = U, \tag{3}$$

$$T(0, t) = T_s(t), \qquad T(y, 0) = T_\infty , \tag{4}$$

where U is the free stream velocity, T_∞ the free stream temperature, and T_s the plate surface temperature.

The problem defined by Eqs. (1) and (3) for the velocity diffusion is well-known, the solution being

$$u = U erf[y/2(\nu t)^{1/2}]. \tag{5}$$

Substituting (5) into (2), we have

$$\frac{\partial T}{\partial t} - a \frac{\partial^2 T}{\partial y^2} = \frac{U^2}{\pi Cp} \frac{\exp(-y^2/2\nu t)}{t}, \tag{6}$$

where $a = k/\rho Cp$. The solution to (6) with the boundary and initial conditions (4) is obtained by the method of heat sources and sinks and the result is:

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$$T - T_\infty = 2(\pi)^{-1/2} \int_{y/2(at)^{1/2}}^{\infty} [T_s(t - y^2/4a\beta^2) - T_\infty] \exp(-\beta^2)\, d\beta$$

$$+ \frac{U^2}{\pi C p} \int_0^\infty d\lambda \int_0^t \frac{\exp(-\lambda^2/2\nu\tau)}{\tau} \frac{\exp\left[-\frac{(y-\lambda)^2}{4a(t-\tau)}\right] - \exp\left[-\frac{(y+\lambda)^2}{4a(t-\tau)}\right]}{[4a\pi(t-\tau)]^{1/2}}\, d\tau. \tag{7}$$

The heat flow through a unit surface of the flat plate, Q, is given by the derivative of (7) evaluated at the surface of the plate:

$$Q = k \frac{\partial T}{\partial y}\bigg|_{y=0} = -k(\pi a t)^{-1/2}[T_s(0) - T_\infty] - k(\pi a)^{-1/2} \int_0^t (t-\tau)^{-1/2} T_s'(\tau)\, d\tau$$
$$+ k(\pi a t)^{-1/2} r(\sigma) U^2/2Cp, \tag{8}$$

where the recovery factor, $r(\sigma)$, is given by:

$$r(\sigma) = \frac{4}{\pi}\left(\frac{\sigma/2}{|1 - \sigma/2|}\right)^{1/2} \begin{cases} \tan^{-1}\left(\frac{1-\sigma/2}{\sigma/2}\right)^{1/2}; & \sigma < 2 \\ \log\,[(\sigma/2)^{1/2} + (\sigma/2 - 1)^{1/2}]; & \sigma > 2 \end{cases} \tag{9}$$

and $\sigma = \nu/a =$ Prandtl number. Equation (8) can be written more compactly in the form of a Stieltjes integral as follows:

$$Q = -k(\pi a t)^{-1/2} \int_0^t (1 - \tau/t)^{-1/2}\, d[T_s(\tau) - T_e], \tag{10}$$

where

$$T_e = T_\infty + r(\sigma) U^2/2Cp. \tag{11}$$

The surface-temperature variation to give a prescribed heat flow variation can be obtained by inverting the Abel integral Eq. (10); this gives

$$T_e - T_s(t) = (a/\pi)^{1/2} k^{-1} \int_0^t (t-\tau)^{-1/2} Q(\tau)\, d\tau. \tag{12}$$

For a constant rate of heat flow this reduces to

$$T_e - T_s(t) = 2Qk^{-1}(at/\pi)^{1/2}. \tag{12a}$$

Emmons has already given this latter solution for constant heat flow to the plate [1].

By replacing t by x/U in (1) and (2), we have the linearized boundary-layer equations for steady, viscous, incompressible flow past a flat plate (sometimes called the Rayleigh equations). The boundary conditions (3) and (4) become

$$u(0, x) = 0, \qquad u(y, 0) = U, \tag{13}$$

$$T(0, x) = T_s(x), \qquad T(y, 0) = T_\infty \tag{14}$$

These imply a semi-infinite flat plate with an arbitrary surface temperature $T_s(x)$, where x is the distance from the leading edge in the direction of flow. Therefore, an approximation to the heat flow to a semi-infinite flat plate in a steady flow of velocity U,

and with an arbitrary surface-temperature distribution $T_s(x)$, can be obtained by replacing t by x/U in Eq. (10); this gives

$$Q = -\pi^{-1/2}k(U/\nu x)^{1/2}\sigma^{1/2} \int_0^x (1 - \xi/x)^{-1/2}\, d[T_s(\xi) - T_e] \tag{15}$$

which, for constant surface temperature reduces to the familiar form

$$Q = \pi^{-1/2}k(U/\nu x)^{1/2}\sigma^{1/2}(T_e - T_s). \tag{15a}$$

Lighthill [2] has given an expression for heat transfer to an arbitrary two-dimensional surface in terms of the skin friction and temperature along the surface, by using the Fage and Falkner linear approximation of the boundary-layer profile and neglecting the viscous dissipation terms in the energy equation. For the case of the flat plate, his result is:

$$Q = -0.339k(U/\nu x)^{1/2}\sigma^{1/3} \int_0^x [1 - (\xi/x)^{3/4}]^{-1/3}\, d[T_s(\xi) - T_\infty]. \tag{16}$$

He argued that the effect of viscous dissipation is taken care of by replacing T_∞ in the above expression by $T_\infty + \sigma\frac{1}{2}U^2/2Cp$, the boundary-layer equilibrium temperature. If, following the suggestion of Lewis and Carrier [3], we replace U by $0.35U$ to approximate a mean convective velocity in the boundary layer the constant multipliers of Eqs. (15) and (16) are nearly equal.

The differential equations used here and by Emmons in [1] apply to a fictitious fluid of constant pressure and density, but variable temperature. The equations are really of interest only because the compressible fluid boundary layer equations can be reduced to their form by the Von Mises transformation and the assumptions that μ is proportional to the temperature and σ is constant (see for example ref. [4]). If enthalpy is used as the independent variable instead of temperature, no additional assumption need be made on the variation of the specific heat with temperature. Then the only change in the previous differential equations is to replace y by η where

$$\eta = \int_0^y \frac{\rho}{\rho_\infty}\, dy$$

The expressions for heat transfer rate are unchanged, although as they stand C_P must be assumed constant.

Bibliography

1. H. W. Emmons, *Note on aerodynamic heating*, Q. Appl. Math. **8**, 402-405 (1951).
2. M. J. Lighthill, *Contributions to the theory of heat transfer through a laminar boundary layer*, Proc. of Roy. Soc. (A) **202**, 359-377 (1950).
3. J. A. Lewis and G. F. Carrier, *Some remarks on the flat plate boundary layer*, Q. Appl. Math. **7**, 228-233 (1949).
4. C. R. Illingworth, *Unsteady laminar flow of gas near an infinite flat plate*, Proc. Cambridge Phil. Soc. **46**, 603-613 (1950).

A NOTE ON THE DAMPING IN ROLL OF A CRUCIFORM WINGED BODY*

By JOHN W. MILES** (*Fulbright Lecturer, Auckland University College, Auckland, New Zealand*)

1. Introduction. In the following we apply the slender body theory of Ward[1] to the calculation of the damping in roll of a slender body of revolution of radius a, bearing cruciform wings of *total* span $2b$.

It is required to find a solution, $\phi(s, x, y)$, to Laplace's equation satisfying the boundary conditions

$$\phi_y(s, x, y) \mid_{y=0} = px, \qquad a < | x | < b, \tag{1a}$$

$$\phi_x(s, x, y) \mid_{x=0} = -py, \qquad a < | y | < b, \tag{1b}$$

$$\phi_r(s, r \cos \theta, r \sin \theta) \mid_{r=a} = 0, \tag{1c}$$

where (s, x, y) are a set of right handed, Cartesian coordinates with s measured positive downstream from the body nose, and p is the angular velocity about the s axis. The rolling moment is then given by

$$N = -\rho U \oint \phi(l, x, y)(x\, dx + y\, dy), \tag{2}$$

where the integral is taken around the cross section at the trailing edge ($s = l$), the latter being assumed to be straight and transverse the free stream. Whereas in Eq. (1) a and b are functions of s, exhibiting a monotonic increase therewith, we hereafter refer only to their values at $s = l$.

2. Solution for potential. The conformal transformation

$$(z + a^2/z)^2 - (b + a^2/b)^2 = (\zeta - c^2/\zeta)^2, \tag{1}$$

$$c^2 = \tfrac{1}{2}(b^2 + a^4/b^2) \tag{2}$$

maps the profile of the cruciform winged body in the $z(=x + iy)$ plane on the circle $| \zeta | = c$ in the ζ plane, the wings appearing as four, symmetrically disposed arcs of subtended angle $2\varphi_0$, where

$$\cos 2\varphi_0 = (a/c)^2 = 2(a/b)^2[1 + (a/b)^4]^{-1}. \qquad (3$$

Transforming the boundary conditions (1.1) and carrying out the solution to the then classical problem, we obtain the potential on $\zeta = ce^{i\varphi}$ in the form

$$\phi = -(2pc^2/\pi) \sin 4\varphi \int_0^{\varphi_0} [(\cos 2\psi - \cos 2\varphi_0) + (\cos^2 2\psi - \cos^2 2\varphi_0)^{1/2}] \cdot (\cos 4\psi - \cos 4\varphi)^{-1}\, d\psi. \tag{4}$$

Carrying out the required integrations in Eq. (4), we have

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$$\phi = -(pc^2/2\pi)\left\{\cos 2\varphi \ln \left| \frac{\tan (\varphi + \varphi_0)}{\tan (\varphi - \varphi_0)} \right| - \cos 2\varphi_0 \ln \left| \frac{\sin 2(\varphi + \varphi_0)}{\sin 2(\varphi - \varphi_0)} \right| \right.$$

$$+ \mathbf{K}(\sin 2\varphi_0) \sin 4\varphi + 2[\mathbf{E}(\sin 2\varphi_0)\mathbf{F}(\sin^{-1} (\sin 2\varphi/\sin 2\varphi_0), \sin 2\varphi_0)$$

$$\left. - \mathbf{K}(\sin 2\varphi_0)\mathbf{E}(\sin^{-1} (\sin 2\varphi/\sin 2\varphi_0), \sin 2\varphi_0)](\sin^2 2\varphi_0 - \sin^2 2\varphi)^{1/2}\right\}, \tag{5}$$

where $\mathbf{K}$ and $\mathbf{E}$, with single arguments, denote complete elliptic integrals of the first and second kinds; respectively, and $\mathbf{F}$ and $\mathbf{E}$, with two arguments, denote the corresponding incomplete integrals of modulus $\sin 2\varphi_0$ and amplitude $\sin^{-1} (\sin 2\varphi/\sin 2\varphi)$. For the special case of no body ($a = 0$, $\varphi_0 = \pi/4$) Eq. (5) reduces to

$$\phi = -(pb^2/2\pi) \cos 2\varphi \ln \left| \frac{1 + \sin 2\varphi}{1 - \sin 2\varphi} \right| \qquad (a = 0) \tag{6}$$

3. Rolling moment. The integrals subsequent to the substitution of Eq. (2.5) in Eq. (1.2) appear to be intractable, and it is expedient to proceed by substituting instead Eq. (2.4), which leads to

$$N/N_0 = (2/\pi)^2[1 + (a/b)^4]^2 \sum_1^\infty (I_n^2/n), \tag{1}$$

$$N_0 = -\pi\rho U p b^4/4, \tag{2}$$

$$I_n = \int_0^{\varphi_0} \sin (4n\varphi)d[\cos 2\varphi + (\cos^2 2\varphi - \cos^2 2\varphi_0)^{1/2}]. \tag{3}$$

The reference moment, N_0, is twice that acting on a single wing of (small) span $2b$, so that N/N_0 represents an overall interference factor. For values of (a/b) near unity the convergence of Eq. (1) is poor, but the potential may be expanded in powers of φ_0 to obtain

$$N/N_0 = 8[1 - (a/b)]^2\{1 - 1.57[1 - (a/b)]\} + O\{[1 - (a/b)]^4\} \tag{4}$$

For the special case $a = 0$, the substitution of Eq. (2.6) in Eq. (1.2) yields an interference factor of $(8/\pi^2)$, which may be checked by summing Eq. (1), viz.

$$N/N_0 = (2/\pi)^2 \sum_1^\infty \left[\left(\frac{4n}{4n^2 - 1}\right)^2\!/n\right] = 8/\pi^2 \tag{5}$$

in agreement with the result obtained by Adams.[2] We remark that the first three terms in Eq. (5) yield the correct result to better than 2%, implying a satisfactorily rapid convergence of Eq. (1) for small (a/b).

More complete numerical results are to be given in a forthcoming NOTS report,[3] representing work supported by the Office of Naval Research.

References

(1) G. N. Ward, *Supersonic flow past slender pointed bodies*, Q. J. Mech. Appl. Math. **2**, 75–97 (1949).

(2) G. J. Adams, *Theoretical damping in roll and rolling effectiveness of slender cruciform wings*, NACA TN 2270 (1951).

(3) J. W. Miles, *On the damping in roll of a slender cruciform wing body*, USNOTS Report, Inyokern, California, (1951).

THE ELASTIC AXES OF A ONE-MASS ELASTICALLY SUPPORTED SYSTEM*

By J. J. SLADE, JR.

When an elastically supported rigid body is subjected to the action of a rectilinear sinusoidal force, the resulting steady motion generally consists of rectilinear and torsional oscillations with frequency equal to that of the exciting force. It is desired to determine the location of the exciting force so that the torsional oscillations are suppressed or, at least, so that the amplitude of these oscillations is reduced to a minimum. The problem arises, for example, in connection with unbalanced machines on elastic foundations, as well as in investigations of the dynamic characteristics of elastically supported rigid assemblies by means of induced vibrations.

The two-dimensional problem has been considered under simplifying conditions.[1] The three-mass mechanical oscillator[2] that produces a force the axis of which may be made to coincide with any line in a fixed plane, when the oscillator is in a fixed position, presents problems that require an extension of existing theory. The present investigation deals with the general case.

We consider a rigid body of mass m that can move freely under general linear elastic constraints with linear damping. Let r be the displacement of its center of gravity with respect to its position in static equilibrium. Since only small oscillations are considered, elastic and damping reactions may be taken to be fixed to a moving frame with origin at r.

Let $\Phi + \epsilon\Phi_0$ and $\Psi + \epsilon\Psi_0$ be dual dyadics such that $-(\Phi + \epsilon\Phi_0)\cdot r$ is the motor[3] of the elastic suspension and $-(\Psi + \epsilon\Psi_0)\cdot r'$ that of the damping system, due to a rectilinear displacement, the prime denoting differentiation with respect to time.

Finally let f be the exciter force and p a point on its line of action. It should be noted that in all cases considered the exciter is rigidly connected to, and forms part of the system. The exciter force is strictly fixed in the moving frame.

The motion of the center of gravity of the body is governed by the equation

$$mr'' + \Psi\cdot r' + \Phi\cdot r = f. \tag{1}$$

There is also the moment

$$c = p \times f - (\Phi_0\cdot r + \Psi_0\cdot r') \tag{2}$$

that tends to produce torsional oscillations.

If the angular frequency of the exciter force is ω, we may write

$$f, r, c = (F, R, C)e^{i\omega t}$$

where

$$(-m\omega^2 I + i\omega\Psi + \Phi)\cdot R = F \tag{3}$$

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[1]See, for example, E. Rausch, *Machinenfundamente und andere dynamische Bauaufgaben*, Ch. III, V.D.I., Berlin, 1936.

[2]R. K. Bernhard, *Study on mechanical oscillators*, Proc. Am. Soc. Test. Materials **29**, 1016-1036 (1949).

[3]L. Brand, *Vector and tensor analysis*, Ch. II, J. Wiley & Sons, New York, 1947.

and

$$C = p \times F - (\Phi_0 + i\omega\Psi_0)\cdot(-m\omega^2 I + i\omega\Psi + \Phi)^{-1}\cdot F$$
$$= p \times F + (\Gamma + i\Lambda)\cdot F. \tag{4}$$

Our problem is to determine under what conditions, if any, we can make $C = 0$ with $F \neq 0$.

An axis with which the line vector $F + \epsilon p \times F$ must coincide to satisfy these conditions fully is here called an elastic axis[4] of the system. An axis of fixed direction with which the line vector must coincide to make $| C | \neq 0$ a minimum will be called a quasi-elastic axis.

Suppose first that the system is conservative, so that $\Psi + \epsilon\Psi_0 = 0$. If oscillations about the axis of the free vector a are suppressed, then $a \cdot C = 0$; or, since in the conservative case $\Lambda = 0$,

$$a \cdot p \times F + a \cdot \Gamma \cdot F = 0. \tag{5}$$

Now, the left hand member of this equation is the moment of the fixed motor $a + \epsilon a \cdot \Gamma$ about the axis $F + \epsilon p \times F$ Whence:

Oscillations about an axis a are suppressed when the line vector $F + \epsilon p \times F$ coincides with a line of the null system of the motor $a + \epsilon a \cdot \Gamma$.

Let e_1 , e_2 , e_3 be unit vectors in the directions of the principal axes of the elastic suspension. In this presentation the diagonal elements of Γ are zero and $e_k \cdot e_k \cdot \Gamma = 0$, so that the motor $e_k + \epsilon e_k \cdot \Gamma$ is a line vector. We therefore have the following results.

1. Rotational oscillations about a principal axis are suppressed when the line of action of the exciter force coincides with a line of the special linear line complex the axis of which is $e_k + \epsilon e_k \cdot \Gamma$.

2. Rotational oscillations about two principal axes are simultaneously suppressed when the exciter force coincides with a line of the linear congruence, the directrices of which are $e_j + \epsilon e_j \cdot \Gamma$, $j = k, l$.

3. The elastic axes of the system are the lines of the regulus the directrices of which are $e_k + \epsilon e_k \cdot \Gamma$, $k = 1, 2, 3$.

When the system is not conservative, the following equation must be added to (5):

$$a \cdot \Lambda \cdot F = 0. \tag{6}$$

4. Oscillations about a are suppressed when the force coincides with a line of the null system of $a + \epsilon a \cdot \Gamma$ that is perpendicular to the fixed couple vector $a \cdot \Lambda$.

Assuming that the principal axes of Ψ coincide with those of Φ, as they generally do in practical cases, we may further state.

1. Rotational oscillations about a principal axis e_k are suppressed when $F + \epsilon p \times F$ is a line of the plane through $e_k + \epsilon e_k \cdot \Gamma$ perpendicular to $e_k \cdot \Lambda$.

2. When $F + \epsilon p \times F$ is the line of intersection of two such planes oscillations are simultaneously suppressed about the corresponding two principal axes.

In general the non-conservative system possesses no elastic axes. Since Γ and Λ are constants, when ω is fixed, we see from Equation 4) that, if F is held fixed, $| C |$ is a minimum when p is so determined that

$$p \times F + \Gamma \cdot F = 0. \tag{7}$$

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[4]Rausch, *loc. cit.*, uses the terms *elastiche Hauptachse* and *elasticher Mittelpunkt.*

This leads to the following result: The quasi-elastic axes of the system are the lines of the regulus the directrices of which are $e_k + \epsilon e_k \cdot \Gamma$, $k = 1, 2, 3$.

The traces of the directrices $e_k + \epsilon e_k \cdot \Gamma$ on the principal planes have been called elastic centers. The locus of an elastic center starts, with $\omega = 0$, at a point that depends on the parameters of the elastic suspension and ends at the center of gravity ($\omega = \infty$).

In the conservative case this locus is the outside section of a hyperbola, the inside section corresponding to $\omega^2 < 0$. The locus is a 4th degree algebraic curve in the non-conservative case. The reduced system in which one reaction goes through the center of gravity and the other two lie in a plane through this center has been considered in detail.[5]

HEAVY DISK SUPPORTED BY CONCENTRATED FORCES*

By YI-YUAN YU (*Washington University, St. Louis, Mo.*)

Muschelišvili solved the problem of a two-dimensional light disk subjected to an arbitrary number of concentrated forces by means of his method of complex variable [1, 273-274].** When the weight of the disk has to be taken into consideration, the problem may still be solved in a similar way. Muschelišvili's notations will be followed throughout this paper, and only additional ones will be defined as they first occur.

In plane problems including body forces due to gravity, the stress function U may still satisfy the biharmonic equation if it is defined by the following equations:

$$\frac{\partial^2 U}{\partial x^2} = \tau_{yy} - V_1\,, \qquad \frac{\partial^2 U}{\partial y^2} = \tau_{xx} - V_1\,, \qquad -\frac{\partial^2 U}{\partial x\,\partial y} = \tau_{xy} \tag{1}$$

in which V_1 is the body force potential due to gravity and is equal to wy when gravity acts in the negative y-direction [3], w being the specific weight of the material of the body. Hence, the function U may be expressed in terms of two analytic functions as shown by Muschelišvili [2, 284].

The boundary conditions

$$\tau_x = \tau_{xx}\frac{dy}{ds} - \tau_{xy}\frac{dx}{ds}, \qquad \tau_y = \tau_{xy}\frac{dy}{ds} - \tau_{yy}\frac{dx}{ds}$$

however, may be shown to lead to some different result. When stress components given by Eqs. (1) are substituted into these conditions and computations carried out in the same manner as given by Muschelišvili [2, 301-302], the following result is obtained:

$$\varphi_1(z) + z\bar{\varphi}_1'(\bar{z}) + \bar{\psi}_1(\bar{z}) = i\int (\tau_x + i\tau_y)\,ds - \int V_1\,dz$$

If we define

$$f_1 + if_2 = i\int (\tau_x + i\tau_y)\,ds - \int V_1\,dz \tag{2}$$

[5]R. K. Bernhard and J. J. Slade, Jr., *On the elastic center of one-mass plane oscillatory systems* (unpublished). Dynamics Laboratory, Bureau of Engineering Research, Rutgers University.

*Received October 15, 1951.

**The first number in each square bracket refers to the References listed at the end of the paper. The subsequent numbers, if any, refer to the page numbers of the reference.

then the boundary equation for the first fundamental boundary value problem is

$$\varphi_1(z) + z\bar{\varphi}_1'(\bar{z}) + \bar{\psi}_1(\bar{z}) = f_1 + if_2$$

which holds true on the boundary C of the body in the original z-plane. When C is mapped into the unit circle γ in the ζ-plane by means of the function $z = \omega(\zeta)$, the boundary equation becomes

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\bar{\omega}'(\bar{\sigma})}\,\bar{\varphi}'(\bar{\sigma}) + \bar{\psi}(\bar{\sigma}) = f_1 + if_2 \qquad (\text{on } \gamma) \tag{3}$$

Thus, except with a different definition of $f_1 + if_2$, this equation assumes the same form as the one for zero body forces [2, 294].

Modified expressions for stress and displacement components may similarly be derived. Only stress components in curvilinear coordinates are given here:

$$\left.\begin{aligned} \tau_{\rho\rho} + \tau_{\theta\theta} &= 2[\Phi(\zeta) + \bar{\Phi}(\bar{\zeta})] + 2V(\zeta) \\ \tau_{\theta\theta} - \tau_{\rho\rho} + 2i\tau_{\rho\theta} &= \frac{2\zeta^2}{\rho^2\bar{\omega}'(\bar{\zeta})}\,[\bar{\omega}(\bar{\zeta})\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)] \end{aligned}\right\} \tag{4}$$

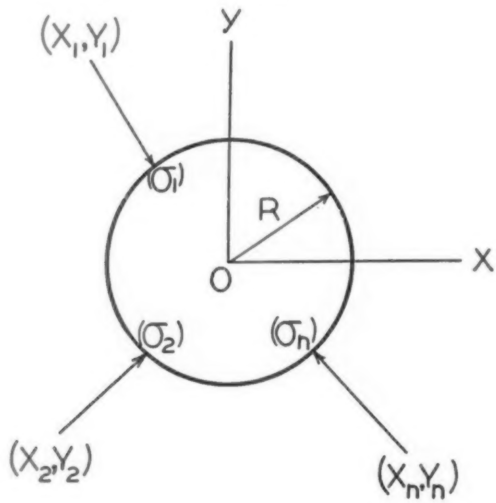
in which

$$V(\zeta) = V_1(\omega(\zeta)) = V_1(z)$$

By comparing with Muschelišvili's original formulas [2, 312], it can be seen that $2V(\zeta)$ is the only additional term due to body forces.

The problem here concerned is that of a heavy disk having radius R supported by an arbitrary number of, say n, concentrated forces (X_1, Y_1), (X_2, Y_2), $\cdots$, (X_n, Y_n) at points on its boundary corresponding to $\sigma_1, \sigma_2, \cdots, \sigma_n$ respectively on the unit circle as shown in the figure. Obviously the supporting forces must satisfy the following conditions:

$$\sum_{k=1}^{n} X_k = 0, \qquad \sum_{k=1}^{n} Y_k = w\pi R^2$$



Since the boundary of the disk is free from stress everywhere except at the supporting points, expression (2) becomes

$$f_1 + if_2 = -\int V_1\, dz = \frac{1}{2}\, iw \int (z - \bar{z})\, dz$$

For a circle of radius R, we have

$$z = \omega(\zeta) = R\zeta$$

hence,

$$f_1 + if_2 = \frac{1}{2}\, iw \int R^2(\sigma - \bar{\sigma})\, d\sigma = \frac{1}{2}\, iwR^2\left(\frac{\sigma^2}{2} - \log \sigma\right)$$

The boundary equation (3) now takes the form

$$\varphi(\sigma) + \sigma\bar{\varphi}'(\bar{\sigma}) + \bar{\psi}(\bar{\sigma}) = \frac{1}{2}\, iwR^2\left(\frac{\sigma^2}{2} - \log \sigma\right) \qquad (\text{on } \gamma) \tag{5}$$

By modifying those obtained by Muschelišvili for the problem of a light disk [1, 274], the two analytic functions for the present problem can be written down as

$$\left.\begin{aligned} \varphi(\zeta) &= -\frac{1}{2\pi} \sum_{k=1}^{n} (X_k + iY_k) \log (\sigma_k - \zeta) + \varphi^0(\zeta) \\ \psi(\zeta) &= \frac{1}{2\pi} \sum_{k=1}^{n} (X_k - iY_k) \log (\sigma_k - \zeta) - \frac{1}{2\pi} \sum_{k=1}^{n} \frac{(X_k + iY_k)\bar{\sigma}_k}{\sigma_k - \zeta} + \psi^0(\zeta) \end{aligned}\right\} \tag{6}$$

in which $\varphi^0(\zeta)$ and $\psi^0(\zeta)$ are functions analytic in the entire region inside γ and have the forms [1, 272]

$$\varphi^0(\zeta) = \alpha_1\zeta + (\alpha_2 + i\beta_2)\zeta^2 + \cdots$$

$$\psi^0(\zeta) = \alpha_0' + i\beta_0' + (\alpha_1' + i\beta_1')\zeta + (\alpha_2' + i\beta_2')\zeta^2 + \cdots$$

The other terms in $\varphi(\zeta)$ and $\psi(\zeta)$ account for the singularities in the solution due to concentrated forces and therefore have the same forms as those for a light disk.

Substituting expressions (6) into Eq. (5) now yields, after simplification,

$$\varphi^0(\sigma) + \sigma\bar{\varphi}^{0\prime}(\bar{\sigma}) + \bar{\psi}^0(\bar{\sigma}) = \frac{iwR^2}{4}\, \sigma^2 - \frac{1}{2\pi} \sum_{k=1}^{n} (X_k - iY_k)\sigma_k\sigma \tag{7}$$

from which $\varphi^0(\zeta)$ and $\psi^0(\zeta)$ can be determined. Following the established procedure of the method, we formulate an integral equation by multiplying Eq. (7) through by $1/2\pi i\, d\sigma/(\sigma - \zeta)$ and integrating around γ. The integrals in the equation thus obtained can be evaluated by means of the theorems developed by Muschelišvili [1, 269]. The result gives

$$\varphi^0(\zeta) + \alpha_1\zeta + 2(\alpha_2 - i\beta_2) + \alpha_0' - i\beta_0' = \frac{iwR^2}{4}\, \zeta^2 - \frac{1}{2\pi} \sum_{k=1}^{n} (X_k - iY_k)\sigma_k\zeta$$

in which the constant terms may be neglected. The constant α_1 is determined by differ-

entiating the rest of the equation with respect to ζ once and setting ζ equal to zero. We have finally

$$\varphi^0(\zeta) = \frac{iwR^2}{4}\zeta^2 - \frac{1}{4\pi}\sum_{k=1}^{n}(X_k - iY_k)\sigma_k\zeta \tag{8}$$

Substituting this back into Eq. (7) and formulating the conjugate of the result, we obtain

$$\psi^0(\sigma) = -\frac{1}{2}iwR^2 + \frac{1}{4\pi}\sum_{k=1}^{n}[(X_k - iY_k)\sigma_k - (X_k + iY_k)\bar{\sigma}_k]\bar{\sigma}$$

Multiplying this through by $1/2\pi i\, d\sigma/(\sigma - \zeta)$ and integrating around γ,

$$\psi^0(\zeta) = -\tfrac{1}{2}iwR^2 \tag{9}$$

Thus the problem is completely solved. The solution consists of $\varphi(\zeta)$ and $\psi(\zeta)$ as given by (6), and $\varphi^0(\zeta)$ and $\psi^0(\zeta)$ are given respectively by (8) and (9).

The problem of a heavy disk resting on a horizontal plane was solved by J. H. Michell [4] and represents a special case of our problem in which

$$n = 1, \qquad X_1 = 0, \qquad Y_1 = w\pi R^2, \qquad \sigma_1 = -i$$

The analytic functions reduce to

$$\varphi(\zeta) = -\frac{iwR^2}{2}\log(\zeta + i) + \frac{iwR^2}{4}\zeta^2 + \frac{wR^2}{4}\zeta$$

$$\psi(\zeta) = -\frac{wR^2}{2}\frac{1}{\zeta + i} - \frac{iwR^2}{2}\log(\zeta + i) - \frac{1}{2}iwR^2$$

The sum of the normal stress components can be computed according to the first of Eqs. (4); thus,

$$\tau_{\rho\rho} + \tau_{\theta\theta} = wR\left(1 - 2\frac{\rho\sin\theta + 1}{\rho^2 + 2\rho\sin\theta + 1}\right)$$

It can readily be shown that both normal stress components vanish at all points on the boundary of the disk except the point of support.

The problem of a heavy disk resting on the ends of its horizontal diameter, as was recently solved by Horvay and Poritsky [5], is another special case in which

$$n = 2, \qquad X_1 = X_2 = 0, \qquad Y_1 = Y_2 = \tfrac{1}{2}w\pi R^2, \qquad \sigma_1 = 1, \qquad \sigma_2 = -1$$

The analytic functions are

$$\varphi(\zeta) = -\frac{iwR^2}{4}\log(\zeta^2 - 1) + \frac{iwR^2}{4}\zeta^2$$

$$\psi(\zeta) = \frac{iwR^2}{2}\frac{1}{\zeta^2 - 1} - \frac{iwR^2}{4}\log(\zeta^2 - 1) - \frac{1}{2}iwR^2$$

The normal stress components at any point on the boundary of the disk except the two at the supports are given by

$$(\tau_{\rho\rho})_{\rho=1} = 0, \qquad (\tau_{\theta\theta})_{\rho=1} = -\frac{wR}{\sin\theta}$$

REFERENCES

1. N. Muschelišvili, *Praktische Lösung der fundamentalen Randwertaufgaben der Elastizitätstheorie in der Ebene für einige Berandungsformen*, ZaMM **13**, 264-282 (1933).
2. N. Muschelišvili, *Recherches sur les problèmes aux limites relatifs à l'équation biharmonique et aux équations de l'élasticité à deux dimensions*, Math. Annalen **107**, 282-312 (1933).
3. S. Timoshenko, *Theory of elasticity*, McGraw-Hill, New York, 1934, pp. 25-26.
4. H. Love, *Mathematical theory of elasticity*, 4th ed., 1927, p. 219.
5. G. Horvay and H. Poritsky, *Gravitational stresses in a disk supported at the ends of the horizontal diameter*, General Electric, Knolls Atomic Power Lab., Schenectady, N. Y., 1951.

REPRESENTATION OF NONLINEAR FIELD FUNCTIONS BY THIELE SEMI-INVARIANTS[1]

BY WILLIAM M. MACDONALD, III,[2] JOHN M. RICHARDSON,[3] AND LEON P. ROSENBERRY[4]

1. Certain nonlinear field functions, possessing the property of space localization of the gradients of the dependent variables, occur in the problem of flame propagation in continuous media. The difficulties encountered in solving the equations of propagation may be considerably diminished by introducing new dependent variables (intrinsically connected with this localization) within a certain region, outside of which the equations may be linearized and treated in a point-wise sense by well-known methods. Specifically, we choose the x-axis as nearly perpendicular to the flame front and define the region as $x_1(y, z) \leqq x \leqq x_2(y, z)$. Taking a state variable, temperature T, for example, we choose the new dependent variables as the κ_ν given by

$$\sum_{\nu=0}^{\infty} (it)^\nu \frac{\kappa_\nu}{\nu!} = \log [(\phi t)] \tag{1.1}$$

where

$$\phi(t) = \int_{x_1}^{x_2} dx e^{itx} \frac{\partial T}{\partial x} \cdot \tag{1.2}$$

These definitions are closely related to the formalism of mathematical statistics. In particular, if one considers $\partial T/\partial x$ to correspond to an unnormalized distribution function with range $(x_1 \leqq x \leqq x_2)$, the κ_ν correspond to the Thiele semi-invariants [1] of $\partial T/\partial x$ and completely describe a given function in the range $(x_1 \leqq x \leqq x_2)$.

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The semi-invariants were chosen (in preference to the moments

$$\int_{x_1}^{x_2} dx\, x^{\nu} \frac{\partial T}{\partial x}, \tag{1.3}$$

for example) because evaluation of the integrals

$$\int_{x_1}^{x_2} dx\, x^{n}\, T^{s} \frac{\partial T}{\partial x} \tag{1.4}$$

led to application of the Fourier inversion formula to the characteristic function (1.2), where the exponential form considerably lessened the difficulties involved. The investigation was extended to include integrals of the form

$$\int_{x_1}^{x_2} dx\, x^{n}\, T_1\, T_2 \cdots T_m \frac{\partial T_{m+1}}{\partial x}. \tag{1.5}$$

2. The characteristic function $\phi(t)$ of $\partial T/\partial x$, in the region $x_1(y, z) \leqq x \leqq x_2(y, z)$ is given by Eq. (1.2). Using the Fourier inversion formula one finds

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-itx}\, \phi(t) = \frac{\partial T}{\partial x}, \qquad x_1 \leqq x \leqq x_2 \tag{2.1}$$

$$= 0, \qquad \text{otherwise,}$$

from which the following expression for T can be obtained if $\partial T/\partial x$ satisfies certain mild restrictions [2]:

$$T(x) - T(x_1) = \left(\frac{i}{2\pi}\right)\int_{-\infty}^{\infty} dt(e^{-itx} - e^{-itx_1})t^{-1}\phi(t)\, dt. \tag{2.2}$$

Writing

$$\phi(t)e^{-itx} = \int_{x_1}^{x_2} d\xi\left(\frac{\partial T}{\partial \xi}\right) \exp\,[it(\xi - x_1)], \tag{2.3}$$

it can be seen that, for $\partial T/\partial x$ bounded, $\phi(t)e^{-itx}$ has no poles or other singularities other than an essential singularity at infinity and that here

$$\phi(t)e^{-itx} \to 0 \text{ as } t \to \infty, \text{ with } -\pi/2 < \arg t < \pi/2.$$

We use this fact to reduce Eq. (2.2) to a more suitable form by writing

$$\int_{-\infty}^{\infty} dt(e^{-itx} - e^{-itx_1})t^{-1}\phi(t) = P\int_{-\infty}^{\infty} dt\phi(t)e^{-itx}t^{-1} - P\int_{-\infty}^{\infty} dt\phi(t)e^{-itx_1}t^{-1}, \tag{2.4}$$

where $P\int_{-\infty}^{\infty} \cdots$ denotes the Cauchy proper value

$$\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right]_{\lambda}^{\lambda}.$$

Contour integration and application of the theory of residues then yields for the last integral on the right of (2.4) the expression

$$P\int_{-\infty}^{\infty} dt\, e^{-itx_1}t^{-1}\phi(t) = i\pi\phi(0). \tag{2.5}$$

If $\mathfrak{J}(x)$ is defined as

$$\mathfrak{J}(x) = T(x) - \tfrac{1}{2}[T(x_1) + T(x_2)], \tag{2.6}$$

then from Eqs. (1.2), (2.2) and (2.5), one obtains

$$\mathfrak{J}(x) = \left(\frac{i}{2\pi}\right) \int_{-\infty}^{\infty} dt\, e^{-itx} t^{-1} \phi(t) \tag{2.7}$$

and

$$\frac{\partial \mathfrak{J}(x)}{\partial x} = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} dt\, e^{-itx} \phi(t). \tag{2.8}$$

In Eqs. (2.7) and (2.8) and henceforth we conventionally denote $P \int_{-\infty}^{\infty}$ by $\int_{-\infty}^{\infty}$.

3. We now apply (2.7) and (2.8) to

$$\phi_s(t) = \int_{-\infty}^{\infty} dx\, e^{itx} \mathfrak{J}^s \frac{\partial \mathfrak{J}}{\partial x}, \tag{3.1}$$

obtaining

$$\phi_s(t) = i^s (2\pi)^{-s-1} \int_{-\infty}^{\infty} dx\, e^{itx} \left[\left\{ \int_{-\infty}^{\infty} dt\, e^{-itx} t^{-1} \phi(t) \right\}^s \int_{-\infty}^{\infty} dt\, e^{-itx} \phi(t) \right]. \tag{3.2}$$

Using independent variables $t_0, t_1, \cdots, t_s$, one can write

$$\phi_s(t) = i^s (2\pi)^{-s-1} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt_s \cdots \int_{-\infty}^{\infty} dt_0 \prod_{r=0}^{s} [t_r^{-1} \phi(t_r)] \phi(t_0) \exp ix\sigma_s,$$

$$\text{where } \sigma_s(t) = t - \sum_{1}^{s} t_q. \tag{3.3}$$

Since $(2\pi)^{-1} \int_{-\infty}^{\infty} dx\, e^{ixu} = \delta(u)$, one can write

$$\phi_s(t) = i^s (2\pi)^{-s-1} \int_{-\infty}^{\infty} dt_s \cdots \int_{-\infty}^{\infty} dt_1 \prod_{r=1}^{s} t_r^{-1} \phi(t_r) \phi(\sigma_s). \tag{3.4}$$

To evaluate

$$\mu_\nu = \int_{-\infty}^{\infty} dx\, x^\nu \mathfrak{J}^s \frac{\partial \mathfrak{J}}{\partial x}, \tag{3.5}$$

one notes that

$$\phi^{(\nu)}(0) = (d^\nu \phi / dt^\nu)_{t=0} = \alpha_\nu = i^\nu \mu_\nu, \tag{3.6}$$

and hence that

$$\phi_s(t) = \sum_{\nu=0}^{\infty} \alpha_\nu t^\nu / \nu!. \tag{3.7}$$

The α_ν cannot be obtained in analytical form but one can easily expand α_ν in a Taylor's series

$$\alpha_\nu(\kappa_0, \kappa_1, \cdots, \kappa_j, \cdots)$$

$$= (\alpha_\nu)_P + \sum_{i=3}^{\infty} \left(\frac{\partial \alpha_\nu}{\partial \kappa_i}\right)_P \kappa_i + \frac{1}{2} \sum_{i=3}^{\infty} \sum_{j=3}^{\infty} \left(\frac{\partial^2 \alpha_\nu}{\partial \kappa_i\, \partial \kappa_j}\right)_P \kappa_i \kappa_j + \cdots \tag{3.8}$$

about the point $P(\kappa_0, \kappa_1, \kappa_2, 0, 0, 0 \cdots)$ in κ-space (κ_i defined by Eq. (1.1)). The coefficients in this expansion are found to be

$$(\alpha_\nu)_P = A^s_{0,0}(-1)^\nu(\kappa_2/2)^{\nu/2}\mathfrak{I}_s H_\nu(\tau_s), \tag{3.9}$$

$$\left(\frac{\partial \alpha_\nu}{\partial \kappa_\sigma}\right)_P = A^s_{\sigma,0}\mathfrak{I}_s\left[S_{\nu,\sigma} + \left(\sum_{p=1}^{s} t_p^\sigma\right)S_{\nu,0}\right], \tag{3.10}$$

and

$$\left(\frac{\partial^2 \alpha_\nu}{\partial \kappa_\sigma\, \partial \kappa_\mu}\right)_P = A^s_{\sigma,\mu}\mathfrak{I}_s\left[S_{\nu,\sigma+\mu} + S_{\nu\sigma}\sum_{p=1}^{s} t_p^\sigma + S_{\nu\mu}\sum_{p=1}^{s} t_p^\mu + S_{\nu 0}\left(\sum_{p=1}^{s} t_p\right)^2\right], \tag{3.11}$$

where the following notation has been introduced:

$$A^s_{\sigma,\mu} = (i/2\pi)^s i^{\sigma+\mu}/\sigma!\mu!,$$

$$\mathfrak{I}_s = e^{-(s+1)\kappa_0}\int_{-\infty}^{\infty} dt_s \cdots \int_{-\infty}^{\infty} dt_1 \prod_{r=1}^{s} [\phi(t_r)/t_r]\phi\left(-\sum_{q=1}^{s} t_q\right) \text{ an operator,}$$

$$S_{\nu\sigma} = [\{\xi^\sigma\phi_P(\xi)/\phi_P(\xi)\}^{(\nu)}]_{t=0}\,,$$

and $H_\nu(\tau_s)$ is the νth Hermite polynomial.

4. Treatment of the equations of hydrodynamics by the moment transformation will clearly involve integrals in which the integrand includes various combinations of the gradient of a dependent variable and some other variable. We must therefore generalize the foregoing treatment to the case where $\mathfrak{J}$ is not merely the temperature but a generalized vector in a space of n dimensions; this fact can be represented symbolically by writing

$$\mathfrak{J} = (\mathfrak{J}_1, \mathfrak{J}_2, \cdots, \mathfrak{J}_n), \tag{4.1}$$

where the product of two vectors $\mathfrak{J}$ and η gives a vector in the product space of $\mathfrak{J}$ and η, or in our notation,

$$\mathfrak{J}\eta = (\mathfrak{J}_i\eta_j)\begin{pmatrix} i = 1, 2, \cdots, n \\ \\ j = 1, 2, \cdots, n \end{pmatrix}. \tag{4.2}$$

Designating by $\psi(t)$ the characteristic function of the component $\mathfrak{J}_i$ of $\mathfrak{J}$, one finds, as in section 2, that from

$$\psi_i(t) = \exp\left\{\sum_{\nu=0}^{\infty} \kappa_{i\nu}\frac{(it)^\nu}{\nu!}\right\} = \int_{-\infty}^{\infty} dx e^{itx}\frac{\partial \mathfrak{J}_i}{\partial x}, \tag{4.3}$$

follows

$$\frac{\partial \mathfrak{J}_i}{\partial x} = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} dt e^{-itx}\psi_i(t), \tag{4.4}$$

and

$$\mathfrak{J}_i = \left(\frac{i}{2\pi}\right)\int_{-\infty}^{\infty} dt t^{-1}e^{-itx}\psi_i(t). \tag{4.5}$$

Here, $\mathfrak{J}_i$ is again the deviation from the average over the infinite range of x of the corresponding dependent variable. The canonical integral

$$\int_{-\infty}^{\infty} dx e^{itx} \mathfrak{J}^m \frac{\partial \mathfrak{J}}{\partial x} \tag{4.6}$$

represents then the characteristic function $\phi(t)$ whose components are given by the equation

$$\phi_{a_1, a_2, \cdots, a_{m+1}}(t) = \int_{-\infty}^{\infty} dx e^{itx} \mathfrak{J}_{a_1} \mathfrak{J}_{a2} \cdots \mathfrak{J}_{a_m} \frac{\partial \mathfrak{J}_{a_{m+1}}}{\partial x}, \tag{4.7}$$

which is easily transformed by (4.5) into the form

$$\phi_{a_1, a_2, \cdots, a_{m+1}}(t) = \left(\frac{i}{2\pi}\right)^m \int_{-\infty}^{\infty} dt_{a_1} \cdots \int_{-\infty}^{\infty} dt_{a_m} \left(\prod_{i=1}^{m} \Psi_{a_i}(t_i)/t_i \right) \psi_{a_{m+1}}(\xi_m) \tag{4.8}$$

where $\xi_m = t - \sum_{q=1}^{m} t_q$. We write, as before,

$$\phi_{a_1, \cdots, a_{m+1}}(t) = \sum_{\nu=0}^{\infty} \alpha_\nu^{a_1 a_2 \cdots a_{m+1}} (it)^\nu / \nu!, \tag{4.9}$$

where

$$\alpha_\nu^{a_1 a_2 \cdots a_{m+1}} = \phi^{(\nu)}_{a_1, a_2, \cdots, a_{m+1}}(0),$$

the coefficients of this series then being found by a Taylor's series expansion of the form

$$\alpha_\nu^{a_1 a_2 \cdots a_{m+1}} = (\alpha_\nu^{a_1 a_2 \cdots a_{m+1}})_P + \exp \left[\left(\sum_{n=0}^{\infty} \sum_{i=1}^{m} \kappa_{a_i n} \frac{\partial}{\partial \kappa_{a_i n}} \right) \left(\alpha_\nu^{a_1 a_2 \cdots a_{m+1}} \right) \right]_P \tag{4.10}$$

about the point in κ-space for which $\kappa_{a_i n} = 0$ for $n \geqq 3$. The first term is given by

$$(\alpha_\nu^{a_1 a_2 \cdots a_{m+1}})_P = \{\phi^{(\nu)}_{a_1, a_2, \cdots, a_{m+1}}(0)\}_P \tag{4.11}$$

which is, from (4.8),

$$(\alpha_\nu^{a_1 \cdots a_{m+1}})_P = \left(\frac{i}{2\pi}\right)^m \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_m \prod_{i=1}^{m} \left(\frac{\Psi_{a_i}(t_i)}{t_i} \right)_P \left[\Psi^{(\nu)}_{a_{m+1}}(\xi_m)_P \right]_{t=0}. \tag{4.12}$$

On carrying out the operations indicated on the last factor in the integrand one finds (see appendix I)

$$(\alpha_\nu^{a_1 \cdots a_{m+1}})_P = C_\nu \int_s d\tau \left(\prod_{i=1}^{m} t_i \right)^{-1} [\exp (ix't - t'At)] H_\nu(\tau_m), \tag{4.13}$$

$$\left(\frac{\partial \alpha_\nu}{\partial \kappa_{a_i \mu}} \right)_P = \left(\frac{C_\nu i^\mu}{\mu!} \right) \int_s d\tau\, t_i^\mu \{\exp (ix't - t'At)\} H_\nu(\tau_m), \qquad i < m + 1, \tag{4.14}$$

$$\left(\frac{\partial \alpha_\nu}{\partial \kappa_{a_{m+1}, \mu}} \right)_P = \left(\frac{C_\nu i^\mu}{\mu!} \right) \int_s d\tau \left[\prod_{j=1}^{m} t_i \right]^{-1} \{\exp (ix't - t'At)\} S_{\sigma\nu} \tag{4.15}$$

and

$$S_{\sigma\nu} = \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \frac{\sigma!(-1)^{\nu-\mu}}{(\sigma - \mu)!} \left(\frac{\kappa_{a_{m+1} 2}}{2} \right)^{(\nu-\mu)/2} H_{\nu-\mu}(\tau_s),$$

where

$$C_\nu = \left(\frac{i}{2\pi}\right)^m (-1)^\nu \left(\frac{\kappa_{a_{m+1},2}}{2}\right)^{\nu/2} \exp \sum_{i=1}^{m+1} \kappa_{i0} \, .$$

The exponential is the matrix representation of the quadric,

$$ix't - t'At = -\frac{1}{2} \sum_{i=1}^{m} (\kappa_{a_i,2} + \kappa_{a_{m+1},2}) t_i^2 - \frac{1}{2} \kappa_{a_{m+1},2} \sum_{p>q=1}^{m} t_p t_q$$

$$+ i\left[\sum_{j=1}^{m} \kappa_{a_j 1} t_j - \kappa_{a_{m+1} 1} \sum_{j=1}^{m} t_j\right],$$

$$\tau_m = \left(\frac{\kappa_{a_{m+1} 2}}{2}\right)^{1/2} \left(\sum_{p=1}^{m} t_p + \frac{i\kappa_{a_{m+1},1}}{\kappa_{a_{m+1},2}}\right),$$

and the integral is over m-dimensional space.

APPENDIX I—Evaluation of $(\alpha_\nu)_P$ and $(\partial\alpha_\nu/\partial\kappa_\sigma)_P$

These integrals (3.9, 3.10, 3.11) all reduce to integrals of the form

$$\int_\tau d\tau \exp\,[-Af(x_1, x_2, \cdots, x_n)] x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

with $\sum m_K$ even if n is odd and vice versa (m_i integers),

$$f(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} x_i^2 + \frac{1}{2} \sum\sum_{i \neq j} x_i x_j \, .$$

Obviously, several integrations by parts would reduce this to an evaluation of a well-known integral. A simpler technique is available. For example, with $s = 2$ we insert the parameters a, b to form:

$$f(a, b) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 (x_1 x_2)^{-1} \exp\,[-\kappa_2(ax_1^2 + x_2^2 + bx_1 x_2)].$$

One then evaluates $\int_0^b db\, \partial f/\partial b$ to find

$$f(a, b) = -2\pi \sin^{-1} (b/2a^{1/2}).$$

Setting $a = b = 1$ in the derivatives of this function then gives the values of the required integrals.

Moments up to the fifth have been tabulated for $s = 1$ and $s = 2$. For $s = 3$ only first terms for moments up to the fourth have been evaluated.

REFERENCES

[1] Harald Cramér, *Methods of mathematical statistics:* Princeton University Press, New Jersey, 185-186 (1946).

[2] G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, Dover Publications, New York, 101-102 (1943).

ON A "PARADOX" IN BEAM VIBRATION THEORY*

By E. H. LEE (*Brown University*)

1. Introduction. Timoshenko[1] discusses the problem of a constant vertical point force P moving with constant velocity v across a uniform elastic beam of length l simply supported at its ends at the same level as shown in Figure 1. The beam is considered to be at rest and in equilibrium when the force commences to move across the span at $t = 0$. The force sets the beam in vertical oscillation, more or less violent depending on the magnitude of the force and its speed of traverse. In general the beam will be in oscillation, involving both kinetic and elastic strain energy, when the force reaches the right hand support. The "paradox" arises concerning the source of this energy, since the vertical force undergoes no net vertical displacement and so might be considered to do no net work.

Timoshenko's explanation consists in considering a different but related problem. He states that we should consider the force to act through a frictionless constraint, so that the force must always be directed normally to the instantaneous direction of the beam at the point of action. This introduces a horizontal component of force on the beam which is prescribed when the motion due to the vertical force has been determined. This horizontal component does work, and Timoshenko shows that in the case of resonance, when the traverse time is half the fundamental period of the beam, this work is almost equal to the energy in the fundamental mode of oscillation, the discrepancy being attributed to the higher modes of oscillation.

Although this explanation seems reasonable from a purely mechanical standpoint, it remains puzzling as to why one can't specify the dynamical problem in terms of a vertical force. The work of Lagrange showed how useful the concept of frictionless constraints can be in dynamics, but this does not imply that one must use them in specifying a problem instead of prescribing forces. The careful discussion of energy relations given below shows that the paradox does not in fact exist even with a prescribed vertical force. Since the same type of difficulty arises occasionally in other problems, it was thought worthwhile to write this note to clarify the issue.

2. Energy relations. This discussion is concerned with the usual linear theory of small lateral vibration of beams. We shall use Timoshenko's notation, which in addition to the quantities specified in Fig. 1 specifies: Young's modulus E, second moment of

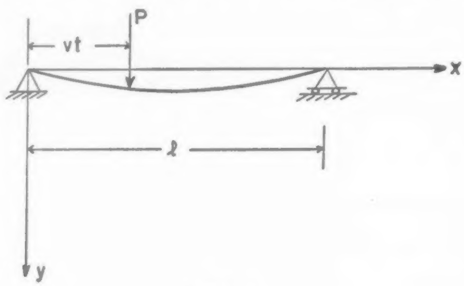

FIG. 1.

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[1]S. Timoshenko, *Vibration problems in engineering*, Van Nostrand, New York, 1937, p. 355.

the section 1, section area A, density γ, and acceleration due to gravity g.

When a force acts on a moving body, the rate of work done on the body is equal to the scalar product of the force and the particle velocity of the body at the point of application. In the present example, therefore, the rate of work done by the vertical force P is $P\ \partial y/\partial t$. However, the vertical component of the velocity of the point of application of the force which moves along the beam is given by a convected derivative:

$$\left(\frac{dy}{dt}\right)_{x=vt} = \frac{\partial y}{\partial t} + v\,\frac{\partial y}{\partial x} \tag{1}$$

Since the force leaves the beam at the level at which it entered the span

$$\int_0^{l/v} \left(\frac{dy}{dt}\right)_{x=vt} dt = 0 \tag{2}$$

The total work done on the beam by the constant vertical force P is:

$$P \int_0^{l/v} \left(\frac{\partial y}{\partial t}\right)_{x=vt} dt \tag{3}$$

which in general will not be equal to zero. In fact it follows directly from (1) and (2) that the work (3) done by the vertical force is equal to the work done by the horizontal component of the constraint force in Timoshenko's modified problem. Thus:

$$P \int_0^{l/v} \left(\frac{\partial y}{\partial t}\right)_{x=vt} dt = -P \int_0^{l/v} \left(\frac{\partial y}{\partial x}\right)_{x=vt} v\, dt. \tag{4}$$

The equality between the work (3) and the energy of the motion induced in the beam follows directly from the differential equation of the motion, and the boundary and initial conditions:

$$\begin{gathered} EI\,\frac{\partial^4 y}{\partial x^4} + \frac{\gamma A}{g}\,\frac{\partial^2 y}{\partial t^2} = P\ \delta(x/vt) \\ y = \frac{\partial^2 y}{\partial x^2} = 0, \qquad x = 0,\, l, \qquad t \geq 0 \\ y = \frac{\partial y}{\partial t} = 0, \qquad t = 0, \qquad 0 \leq x \leq l \end{gathered} \tag{5}$$

where $\delta(x/vt)$ is the Dirac delta function which is non zero at $x = vt$. Multiplying both sides by $\partial y/\partial t$ and integrating with respect to x from 0 to l we obtain, after integration by parts:

$$\frac{\partial}{\partial t}\left[\frac{EI}{2}\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx + \frac{\gamma A}{2g}\int_0^l \left(\frac{\partial y}{\partial t}\right)^2 dx\right] = P\left(\frac{\partial y}{\partial t}\right)_{x=vt} \tag{6}$$

Thus the rate of increase of kinetic plus strain energy is equal to the rate of work as defined in (3).

3. Conclusion. Thus when energy relations are written down carefully, the fact that the vertical force leaves the beam at the level at which it entered the span does not imply no net work done. The reason for misinterpretation is that in most mechanical problems the point of action of an applied force does move with the body on which it is acting. However, there are practical examples where this is not the case. Consider

for example an induction motor. The magnetic force producing the torque moves with the synchronous angular velocity of the alternating current, whereas the rotor speed falls below this by the slip, so that the magnetic force is moving relative to the rotor on which it is producing a torque. The rate of work done on the rotor is the torque multiplied by the rotor speed, and not the torque multiplied by the synchronous speed. In many cases, as in this case, the difference in these two quantities may be lost as mechanical work; in this case it appears as eddy current loss. The energy balance at the point of application is a function of the detailed method of application of the force. A problem, in which a similar difficulty in the use of partial and convected derivatives arose, appeared in the discussion of the motion of a bar containing a hinge moving along its length.[2] In this case the wrong choice involved an apparent paradox: the failure of the momentum principle.

NOTE ON THE LEAST EIGENVALUE OF THE HILL EQUATION

By TOSIO KATO (*University of Tokyo*)

Let us consider the Hill equation

$$y'' + [\lambda + f(x)]y = 0, \qquad -\infty < x < \infty, \tag{1}$$

where $f(x)$ is a real-valued periodic function of period 1 with the Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \exp (2\pi i n x), \qquad \bar{c}_n = c_{-n} .$$

Recently Wintner([5], Eq. (23)) deduced the following inequalities satisfied by the lower limit λ_0 of the spectrum of (1):

$$-c_0 \geqq \lambda_0 \geqq -c_0 - 2 \sum_{n=1}^{\infty} | c_n |^2. \tag{2}$$

Also the question is raised (Putnam [3], p. 314) whether the coefficient 2 on the right-hand side is the least possible value. In the present note we shall show that better estimates do exist. In particular, we shall show that

$$\lambda_0 \geqq -c_0 - \frac{1}{8} \sum_{n=1}^{\infty} | c_n |^2. \tag{3}$$

For this purpose we note that λ_0 is characterized as the least eigenvalue of (1) considered on the *finite* interval $0 \leqq x \leqq 1$ with the periodic boundary conditions

$$y(0) = y(1), \qquad y'(0) = y'(1) \tag{4}$$

(see e.g. Strutt [4], p. 15). Therefore, according to the Ritz variational principle, λ_0 is the minimum value of the expression

$$J[y] = \int_0^1 (y'^2 - fy^2)\, dx \Big/ \int_0^1 y^2\, dx,$$

where y changes over real-valued functions subjected to the conditions (4). Let $z(x)$ be any *periodic* function of period 1 with integrable $z'(x)$. Then we have

$$\int_0^1 (2zyy' + z'y^2)\, dx = [zy^2]_0^1 = 0$$

since both y and z are periodic. Hence

$$\int_0^1 (y'^2 - fy^2)\, dx = \int_0^1 [(y' + zy)^2 + (z' - z^2 - f)y^2]\, dx$$

$$\geqq \text{Min}\,(z' - z^2 - f) \cdot \int_0^1 y^2\, dx.$$

It follows that $J[y] \geqq \text{Min}\,(z' - z^2 - f)$ and this implies

$$\lambda_0 \geqq \text{Min}\,(z' - z^2 - f). \tag{5}$$

Incidentally, this is an adaptation of Wintner's condition ([5], p. 368) to the eigenvalue problem under consideration. By different choices of z we can obtain different lower bounds of λ_0.

First take as z an indefinite integral g of $f_1 \equiv f - c_0$. Then z is certainly periodic and (5) becomes

$$\lambda_0 \geqq \text{Min}\,(-c_0 - g^2) = -c_0 - (\text{Max}\,|\,g\,|)^2. \tag{6}$$

To estimate Max $|\,g\,|$, we introduce the *oscillation* A of g:

$$A = \text{Max}\, g - \text{Min}\, g.$$

Since g is continuous, there are values a, b of x such that $g(a) = \text{Min}\, g$, $g(b) = \text{Max}\, g$. Since g is periodic, we may assume $a < b < a + 1$ without loss of generality. Then we have

$$A = g(b) - g(a) = \int_a^b f_1\, dx = -\int_b^{a+1} f_1\, dx.$$

Hence

$$A \leqq \int_a^b |\,f_1\,|\, dx, \qquad A \leqq \int_b^{a+1} |\,f_1\,|\, dx.$$

Addition of both inequalities and application of the Schwarz inequality give

$$2A \leqq \int_a^{a+1} |\,f_1\,|\, dx = \int_0^1 |\,f_1\,|\, dx \leqq \left[\int_0^1 f_1^2\, dx\right]^{1/2}.$$

Heretofore g has been determined only up to an arbitrary additive constant. If we choose this constant appropriately, we can make $g(b) = -g(a) = A/2$. Then we have Max $|\,g\,| = A/2$ and hence

$$(\text{Max}\,|\,g\,|)^2 \leqq \frac{1}{16}\int_0^1 f_1^2\, dx = \frac{1}{8}\sum_{n=1}^{\infty} |\,c_n\,|^2$$

which, combined with (6), proves (3).

Another estimation of λ_0 is obtained by adjusting the arbitrary constant in g in such a way that

$$g(x) = \sum_{n \neq 0} (2\pi i n)^{-1} c_n \exp(2\pi i n x).$$

Then we have

$$\operatorname{Max} | g | \leqq \pi^{-1} \sum_{n=1}^{\infty} n^{-1} | c_n |$$

and (6) gives

$$\lambda_0 \geqq -c_0 - \pi^{-2}\Big(\sum_{n=1}^{\infty} n^{-1} | c_n | \Big)^2. \tag{7}$$

It will be noted that (3) and (7) are mutually independent. In any case, however, (7) is better than the second inequality of (2), for we have

$$\pi^{-2}(\sum n^{-1} | c_n |)^2 \leqq \pi^{-2} \sum n^{-2} \sum | c_n |^2 = \frac{1}{6} \sum | c_n |^2$$

by the Cauchy inequality. In the same way it is seen that (7) is better than (3) if $c_1 = 0$.

It is not clear whether the coefficient 1/8 in (3) can still be replaced by a smaller one. However, it cannot be made smaller than $1/2\pi^2$ (Cf. Putnam [3], p. 314, where the figure $1/4\pi^2$ is given). In fact, consider the case $f(x) = 2c_1 \cos 2\pi x$ with a real c_1 (Mathieu equation); then the formula of the usual perturbation theory (Courant-Hilbert [1], p. 300) yields easily the expansion

$$\lambda_0 = -(2\pi^2)^{-1} c_1^2 + \cdots$$

which is certainly *convergent* for sufficiently small value of $| c_1 |$ (Kato [2], p. 169).

It will be noted that the lower bounds of λ_0 as given by the formulas (3) and (7) are, though rigorous, not very accurate from the practical standpoint. Especially this is the case when the Fourier coefficients c_n are large, for we have

$$\lambda_0 \geqq -\operatorname{Max} f \geqq -c_0 - 2 \sum_{n=1}^{\infty} | c_n |, \tag{8}$$

as is easily seen by setting $z = 0$ in (5).

For more accurate estimation of λ_0 in individual cases, it is more convenient to use (5) directly. For instance consider the case $f = 2c_1 \cos 2\pi x$ stated above. If we set $z = k \sin 2\pi x$, we have by (5)

$$\begin{aligned}\lambda_0 &\geqq \operatorname{Min} [(2\pi k - 2c_1) \cos 2\pi x - k^2 \sin^2 2\pi x] \\ &= \operatorname{Min} [(k \cos 2\pi x + \pi - k^{-1}c_1)^2 - k^2 - (\pi - k^{-1}c_1)^2] \\ &\geqq -k^2 - (\pi - k^{-1}c_1)^2.\end{aligned}$$

This is true for every k. If we assume $c_1 > 0$ and take $k = c_1^{1/2}$, we obtain

$$\lambda_0 \geqq -2c_1 + 2\pi c_1^{1/2} - \pi^2.$$

It is easily seen that the right-hand side coincides with the asymptotic expansion of λ_0 for $c_1 \to \infty$ up to the order $c_1^{1/2}$ inclusive (Strutt [4], p. 37, Eq. (4), where we have to set $\lambda = \pi^{-2}\lambda_0$, $h^2 = \pi^{-2}c_1$, $m_1 = 1$).

References

1. R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, I, second edition, Berlin, 1931.
2. T. Kato, *On the convergence of the perturbation method*, J. Fac. Sci., Univ. Tokyo, (I) **6**, 145-226 (1951).
3. C. R. Putnam, *On the least eigenvalue of Hill's equation*, Q. Appl. Math. **9**, 310-314 (1951).
4. M. J. O. Strutt, *Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik*, Berlin, 1932.
5. A. Wintner, *On the non-existence of conjugate points*, Amer. J. Math. **73**, 368-380 (1951).

Contents (*Continued from Back Cover*)

Notes:

CONTENTS

CONTENTS (*Continued on Inside Back Cover*)